1. **(20 points)** *Two models of guitar are being produced by a factory. Model A has a labor cost of \$30 and a material cost of \$20, while Model B has a labor cost of \$40 and a material cost of \$30. Over the course of three weeks, you have the following budget allocations available:*

	Week 1	*Week 2*	*Week 3*
Labor	*\$1800*	*\$1750*	*\$1720*
Material	*\$1200*	*\$1250*	*\$1280*

Determine how many guitars of each type should be produced each week.

If we have a 2×1 "production vector" X whose top element is the number of guitars of type A produced, and whose bottom element is the number of guitars of type B produced, we can multiply by the necessary expenses to get the "expense vector" $\begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} X$ whose top element will be the total labor cost and whose bottom element will be the total material cost. Our production each week will thus be dictated by the three products:

$$\begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} X_1 = \begin{bmatrix} 1800 \\ 1200 \end{bmatrix} \qquad \begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} X_2 = \begin{bmatrix} 1750 \\ 1250 \end{bmatrix} \qquad \begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix} X_3 = \begin{bmatrix} 1720 \\ 1280 \end{bmatrix}$$

and these equations are easy to solve once we determine an inverse of our cost matrix; let's do that now using Gauss-Jordan elimination (or any other method you know)

$$\left[\begin{array}{cc|cc} 30 & 40 & 1 & 0 \\ 20 & 30 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{30} & 0 \\ 20 & 30 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{30} & 0 \\ 0 & \frac{10}{3} & -\frac{2}{3} & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{4}{3} & \frac{1}{30} & 0 \\ 0 & 1 & -0.2 & 0.3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0.3 & -0.4 \\ 0 & 1 & -0.2 & 0.3 \end{array}\right]$$

so since $\begin{bmatrix} 30 & 40 \\ 20 & 30 \end{bmatrix}^{-1} = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix}$, we can solve the three systems above easily:

$$X_1 = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1800 \\ 1200 \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1750 \\ 1250 \end{bmatrix} = \begin{bmatrix} 25 \\ 25 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0.3 & -0.4 \\ -0.2 & 0.3 \end{bmatrix} \begin{bmatrix} 1720 \\ 1280 \end{bmatrix} = \begin{bmatrix} 4 \\ 40 \end{bmatrix}$$

and so in the first week, we produce just 60 guitars of type A; on the second,w e should produce 25 guitars of each type, and on the third, we produce 4 of type A and 40 or type B.

2. **(10 points)** Use the substitution method to solve the system of equations $\begin{cases} 7m + 12n = -1 \\ 5m - 3n = 7 \end{cases}$.

We might arbitrarily choose to find a formula for m in the second equation (there are three other equally valid approaches; this is the one with the fewest stray negations to be dealt with). Then:

$$\begin{aligned} 5m - 3n &= 7 \\ 5m &= 3n + 7 \\ m &= \frac{3n + 7}{5} \end{aligned}$$

Substituting back into the first equation,

$$\begin{aligned}
7\left(\frac{3n+7}{5}\right) + 12n &= -1 \\
\frac{21}{5}n + \frac{49}{5} + 12n &= -1 \\
\frac{81}{5}n &= -\frac{54}{5} \\
n &= -\frac{2}{3}
\end{aligned}$$

and then $m = \frac{3n+7}{5} = \frac{-2+7}{5} = 1$.

3. **(10 points)** *For each of the following matrices indicate whether it is in reduced form; if it is not, explain why and indicate the row operation which would put it into reduced form.*

   (a) $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & -1 \end{array}\right]$.
   This matrix is not reduced, because lower pivots are to the left of higher pivots; we can correct it simply by swapping the two rows to get $\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$.

   (b) $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$.
   This matrix is reduced; it has a pivot in the first row and column, and the second row and third column, and a row of zeroes on the bottom.

   (c) $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$.
   The matrix is not reduced; the second row's pivot is not 1. We can divide the second row by 2 to make it correct: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

4. **(15 points)** *Either calculate the inverse of* $\begin{bmatrix} 2 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ *or demonstrate that it has no inverse.*

We perform the steps of Gauss-Jordan reduction on an augmented matrix; other orders of operations are possible, but they should end in the same result:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0.5 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad \left(R_1 \times \tfrac{1}{2} \to R_1\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0.5 & 0 & 0 \\ 0 & 2 & -1 & -0.5 & 1 & 0 \\ 0 & 1 & -1 & -0.5 & 0 & 1 \end{array}\right] \qquad \begin{pmatrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{pmatrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 0.5 & 0 & 0 \\ 0 & 1 & -0.5 & -0.25 & 0.5 & 0 \\ 0 & 1 & -1 & -0.5 & 0 & 1 \end{array}\right] \qquad \left(R_2 \times \tfrac{1}{2} \to R_2\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1.5 & 0.25 & 0.5 & 0 \\ 0 & 1 & -0.5 & -0.25 & 0.5 & 0 \\ 0 & 0 & -0.5 & -0.25 & -0.5 & 1 \end{array}\right] \qquad \begin{pmatrix} R_1 + R_2 \to R_1 \\ R_3 - R_2 \to R_3 \end{pmatrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1.5 & 0.25 & 0.5 & 0 \\ 0 & 1 & -0.5 & -0.25 & 0.5 & 0 \\ 0 & 0 & 1 & 0.5 & 1 & -2 \end{array}\right] \qquad \left(R_3 \times -2 \to R_3\right)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & -1 & 3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0.5 & 1 & -2 \end{array}\right] \qquad \begin{pmatrix} R_1 - 1.5R_3 \to R_1 \\ R_2 + 0.5R_3 \to R_2 \end{pmatrix}$$

so $\begin{bmatrix} 2 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.5 & -1 & 3 \\ 0 & 1 & -1 \\ 0.5 & 1 & -2 \end{bmatrix}$

5. **(15 points)** *Perform the following calculations, or explain why they cannot be performed.*

(a) $\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix}$.

This is a simple matrix product:

$$\begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \cdot 2 + 1 \cdot 8 \\ 2 \cdot 2 + (-3) \cdot 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -20 \end{bmatrix}$$

(b) $3 \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} - 2 \begin{bmatrix} -2 & 6 \\ 5 & -3 \end{bmatrix}$.

This question involves two scalar products and then a matrix subtraction:

$$3 \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} - 2 \begin{bmatrix} -2 & 6 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 3 & 24 \end{bmatrix} - \begin{bmatrix} -4 & 12 \\ 10 & -6 \end{bmatrix} = \begin{bmatrix} 16 & -6 \\ -7 & 30 \end{bmatrix}$$

(c) $\begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 5 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

This calculation cannot be performed: the product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix, which cannot then have a matrix of a different size (3×3) subtracted from it.

6. **(10 points)** *A jar contains pennies (worth 1 cent), nickels (worth 5 cents), and dimes (worth 10 cents). Pennies weigh 2.5 grams, nickels weight 5 grams, and dimes weigh 2 grams. There are 281 coins in the jar, with a value of 1316 cents and a weight of 792 grams. We wonder how many of each coin are in the jar.*

   Convert the situation described into a system of equations which would help us answer our question. Indicate what each of your variables means. You do not need to solve the system.

   We have three unknowns: the number of pennies, nickels, and dimes. We might denote these respectively by the variables x, y, and z. Then we can build a system of three equations describing our three known qualities: total number, total value, and total weight.

$$\begin{cases} x + y + z = 281 \\ x + 5y + 10z = 1316 \\ 2.5x + 5y + 2z = 792 \end{cases}$$

7. **(20 points)** *Use the knowledge that* $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -5 & -12 & 3 \\ -2 & -4 & 1 \\ 2 & 5 & -1 \end{bmatrix}$ *to help solve the following questions.*

   (a) Find a solution to the system of equations $\begin{cases} x - 3y = 3 \\ y + z = -2 \\ 2x - y + 4z = 0 \end{cases}$

   We know that

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

   and thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 & -12 & 3 \\ -2 & -4 & 1 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ -4 \end{bmatrix}$$

   (b) Find a matrix X such that $\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix} X - \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$.

   We reverse the algebraic operations seen to isolate X:

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix} X - \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix} X = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & -12 & 3 \\ -2 & -4 & 1 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \\ -10 \end{bmatrix}$$