1. **(20 points)** *For each of the following propositions, construct a truth table and explicitly identify the proposition as either a tautology, a contradiction, or neither.*

   (a) **(5 points)** $(p \to q) \wedge (p \wedge q)$

p	q	$p \to q$	$p \wedge q$	$(p \to q) \wedge (p \wedge q)$
T	T	T	T	T
T	F	F	F	F
F	T	T	F	F
F	F	T	F	F

   Since this is neither universally true nor universally false, it is neither a tautology nor a contradiction.

   (b) **(5 points)** $p \to (p \vee \neg q)$

p	q	$\neg q$	$p \vee \neg q$	$p \to (p \vee \neg q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

   Since this is universally true, it is a tautology.

   (c) **(10 points)** $[(p \wedge q) \vee (\neg p \wedge r)] \to (q \vee r)$.

p	q	r	$p \wedge q$	$\neg p$	$\neg p \wedge r$	$(p \wedge q) \vee (\neg p \wedge r)$	$q \vee r$	$[(p \wedge q) \vee (\neg p \wedge r)] \to (q \vee r)$
T	T	T	T	F	F	T	T	T
T	T	F	T	F	F	T	T	T
T	F	T	F	F	F	F	T	T
T	F	F	F	F	F	F	F	T
F	T	T	F	T	T	T	T	T
F	T	F	F	T	F	F	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	F	F	F	T

   Since this is universally true, it is a tautology.

2. **(10 points)** *If* $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 5, 7\}$, *and* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, *determine the results of each of the following operations.*

   (a) $A \cup B = \{1, 2, 3, 4, 6, 8\}$.

   (b) $A \cap C = \{3\}$.

   (c) $A' = \{5, 6, 7, 8, 9, 10\}$.

   (d) $B \cap C = \emptyset$.

   (e) $B \cap C' = \{2, 4, 6, 8\}$.

   (f) $B \cup C' = \{1, 2, 4, 6, 8, 9, 10\}$.

   (g) $(A \cup B)' = \{5, 7, 9, 10\}$.

(h) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

(i) $B \cup C = \{2, 3, 4, 5, 6, 7, 8\}$.

(j) $B' \cup C' = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

3. **(40 points)** *A special diet for laboratory animals must contain at least 850 units of vitamins, 800 units of minerals, and 1150 calories. Two feed mixes are available: the light mix, of which each gram costs \$0.04 and contains 2 units of vitamins, 2 units of minerals, and 4 calories; and the heavy mix, of which each gram costs \$0.09 and contains 5 units of vitamins, 4 units of minerals, and 5 calories. How much of each mix should you purchase to minimize the cost?*

(a) **(10 points)** *Formulate a linear programming problem describing the scenario above. Identify what your variables represent, and clearly label constraints and the objective function.*

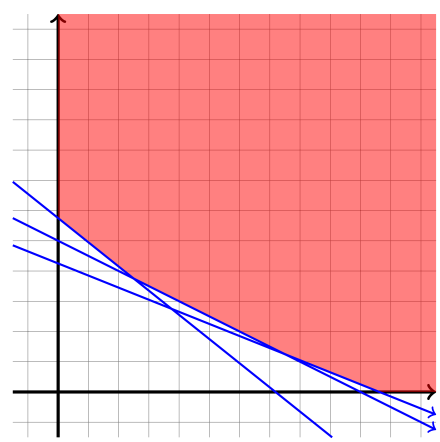
There are two variables: the quantity (in grams) which we buy of the light and heavy mixes; let us label them respectively by x and y.

We have three constraints: we need to produce a feed mix with at least 850 units of vitamins, 800 units of minerals, and 1150 calories. Since a mix of x grams of light mix and y grams of heavy mix provides $2x + 5y$ vitamins, $2x + 4y$ minerals, and $4x + 5y$ calories, our constraints are thus, together with the non-negativity conditions:

$$\begin{cases} 2x + 5y \geq 850 \\ 2x + 4y \geq 800 \\ 4x + 5y \geq 1150 \\ \quad x, y \geq 0 \end{cases}$$

The objective is to minimize the cost, which will be $0.04x + 0.09y$.

(b) **(20 points)** *Solve the above linear programming problem.*



There are four points on the periphery of this region. Two are the intersection of $2x+5y = 850$ with the x-axis and $4x + 5y = 1150$ with the y-axis; these are $(425, 0)$ and $(0, 230)$ respectively. The other two are intersection points of the pairs

$$\begin{cases} 2x + 5y = 850 \\ 2x + 4y = 800 \end{cases}$$

and

$$\begin{cases} 2x + 4y = 800 \\ 4x + 5y = 1150 \end{cases}$$

which are $(300, 50)$ and $(100, 150)$ respectively.

Now we test all four of them to find the minimum:
$0.04 \times 425 + 0.09 \times 0 = 17$
$0.04 \times 300 + 0.09 \times 50 = 16.50$
$0.04 \times 100 + 0.09 \times 150 = 17.50$
$0.04 \times 0 + 0.09 \times 230 = 20.70$

So it is cheapest to use 300 grams of the light mix and 50 of the heavy.

(c) **(10 points)** *The price of the heavy mix decreases to \$0.06 per gram. How does this change the solution you determined above?*

We simply redo our very last step with the new objective function $0.04x + 0.06y$:
$0.04 \times 425 + 0.06 \times 0 = 17$
$0.04 \times 300 + 0.06 \times 50 = 15$
$0.04 \times 100 + 0.06 \times 150 = 13$
$0.04 \times 0 + 0.06 \times 230 = 13.80$

So under these circumstances, buying 100 grams of the light mix and 150 grams of the heavy is the cheapest option.

4. **(15 points)** *Determine the number of ways to do each of the following things:*

(a) **(3 points)** *select one course from each field of study if there are 2 courses available in history, 3 in science, 2 in math, 2 in philosophy, and 1 in English.*

This is a composition of several individual selection procedures: one history course is selected from among 2 possibilities, one science course from among 3 possibilities, one math course from among 2 possibilities, one philosophy course from among 2 possibilities, and one English course from among 1 possibility. Thus, there are $2 \times 3 \times 2 \times 2 \times 1 = 24$ possible selections of a course schedule.

(b) **(3 points)** *select one course* in total *if there are 2 courses available in history, 3 in science, 2 in math, 2 in philosophy, and 1 in English.*

This is a single selection from among $2 + 3 + 2 + 2 + 1$ possibilities, so there are 10 possibilities in total.

(c) **(4 points)** *select a three-digit number where all three digits are different and none of the digits is zero.*

The first digit could be any of 9 different digits. The second could be any of those 9 except the one chosen, so there are 8 possibilities. The third can be any nonzero digit except the two chosen, so there are 7 possibilities. The selection process as a whole can thus be completed $9 \times 8 \times 7 = 504$ different ways.

(d) **(5 points)** *select four committee members from a group of 9 people.*

This is a combination of four things chosen from a set of 9 things, so there are $C_{9,4}$ ways to do it. This evaluates to $\frac{9!}{5!}4! = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126$.

5. **(15 points)** *Determine the probability of each of the following events:*

(a) **(8 points)** *Three cards drawn from a standard 52-card deck are all spades.*

The sample space is the set of all possible three-card draws, of which there are $C_{52,3} = \frac{52\times51\times50}{3\times2\times1} = 22100$. The event in question occurs when three spades are drawn, which, since there are 13 spades in total, can happen in any of $C_{13,3} = 286$ ways, so the probability is $\frac{286}{22100} = \frac{11}{850} \approx 1.29\%$.

(b) **(7 points)** *Rolling two fair 6-sided dice results in a sum of 7 or 11.*

The sample space is the set of all possible rolls of the two dice, which is a set of 36 equally likely pairs. Of those pairs, 6 of them add up to 7, specifically $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$, while only two of them add up to 11, specifically $(5,6)$ and $(6,5)$. Thus 8 of the equally likely 36 elements of the sample space are in our event, and it has probability $\frac{8}{36} = \frac{2}{9} \approx 22\%$.