1. **(8 points)** *For each of the following matrices, either state that the given matrix is in reduced form or explain why it is not and indicate which row operation would transform it into reduced form.*

   (a) **(4 points)** $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 4 \end{array}\right]$.

   This is not in reduced form because there is a nonzero row below a row of all zeroes. To fix this, we can swap the second and third rows, getting a result of $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

   (b) **(4 points)** $\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$.

   This is not in reduced form because there is a nonzero entry in the same column as a pivot: specifically, the 1 in the second row and third column is a pivot, and the -2 directly above it is a violation of reduced form. To fix this, we can add twice the third row to the second, getting a result of $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$.

2. **(10 points)** *Use row operations to change the matrix* $\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right]$ *into reduced form.*

   We see below the series of row operations to reduce the matrix:

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right] \qquad \left(R_1 \cdot \tfrac{1}{2} \to R_1\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 3 & -7 & 2 \end{array}\right] \qquad \left(\begin{array}{l} R_2 - 3R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -5 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -7 & 2 \end{array}\right] \qquad \left(R_2 \cdot \tfrac{1}{3} \to R_2\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \qquad \left(\begin{array}{l} R_1 - 2R_2 \to R_1 \\ R_3 - 3R_1 \to R_3 \end{array}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \left(R_3 \cdot (-1) \to R_3\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \left(\begin{array}{l} R_1 - 2R_2 \to R_1 \\ R_3 - 3R_1 \to R_3 \end{array}\right)$$

3. **(2 points)** *Use your work on the previous question to solve the system of equations*

$$\begin{cases} 2x_1 + 4x_2 - 10x_3 = -2 \\ 3x_1 + 9x_2 - 21x_3 = 0 \\ x_1 + 5x_2 - 12x_3 = 1 \end{cases}.$$

We know that the above system is associated with the matrix $\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right]$. In the previous question we showed that $\left[\begin{array}{ccc|c} 2 & 4 & -10 & -2 \\ 3 & 9 & -21 & 0 \\ 1 & 5 & -12 & 1 \end{array}\right]$ is row-equivalent to $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$, so this system has the same solutions as the system described by the latter matrix. However, since this matrix simply describes the equalities $x_1 = -2$, $x_2 = 3$, and $x_3 = 1$, that triple is the unique solution to the original system, which we might also write as $(-2, 3, 1)$.