Systems of Two Equations in Two Variables

MATH 107: Finite Mathematics

University of Louisville

January ~~6~~ 8, 2014

What is a system of linear equations?

A *linear equation in two variables* is an equation of the form $ax + by = c$, with numbers replacing a, b, and c. Our *variables* are x and y.

A *system of two linear equations* is a pair of equations linear equations, conventionally paired with a curly-brace, as such:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

with numbers replacing a, b, c, d, e, and f.

A *solution* of a system of linear equations is a pair (x, y) of values which makes **both** equations true.

Example and Verification

Here is a system of equations:

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases}$$

Is $(15, 1)$ a solution of this system? No! Even though $15 - 4 \cdot 1 = 11$, $2 \cdot 15 + 1 \neq 2$.

Is $(3, -2)$ a solution of this system? Yes! Both $3 - 4(-2) = 11$ and $2 \cdot 3 + (-2) = 4$.

But how could we know to check $(3, -2)$ in the first place?

Applications

Several questions are solvable with the use of systems of linear equations:

- I have 18 coins, each of which is a penny or a nickel. My collection is worth 42 cents. How many of each coin do I have?
- A museum wants to admit two adults and a child for \$20, and one adult and three children for \$22.50. What admission rates should they charge?
- You want to buy a kilogram of bulk candy for \$8. Caramels weigh 10 grams and cost 5 cents apiece. Chocolates weigh 5 grams and cost 7 cents apiece. How many of each candy should you buy?

Application #1: Coins

Question

I have 18 coins, each of which is a penny or a nickel. My collection is worth 42 cents. How many of each coin do I have?

Two unknown quantities: number of pennies, and number of nickels. Let's denote them x and y.

So we have $x + y$ coins in total, with a value of $1x + 5y$ cents. Thus we'll want a solution to the system of equations:

$$\begin{cases} x + y = 18 \longleftarrow \text{number of coins} \\ x + 5y = 42 \longleftarrow \text{value of coins} \end{cases}$$

Application #2: Museum admissions

Question

A museum wants to admit two adults and a child for \$20, and one adult and three children for \$22.50. What admission rates should they charge?

Two unknown quantities: cost of adult admission in dollars, and cost of child admission in dollars.
Let's denote them x and y.

So two adults and a child cost $2x + y$ dollars, and one adult and three children cost $x + 3y$ dollars. Thus we'll want a solution to the system of equations:

$$\begin{cases} 2x + y = 20 \quad\; \longleftarrow \text{cost of 2 adults and a child} \\ x + 3y = 22.50 \longleftarrow \text{cost of an adult and 3 children} \end{cases}$$

Application #3: Candy purchases

Question

You want to buy 1000 grams of bulk candy for 800 cents. Caramels weigh 10 grams and cost 5 cents apiece. Chocolates weigh 5 grams and cost 7 cents apiece. How many of each candy should you buy?

Two unknown quantities: quantity of caramels purchased, and quantity of chocolates purchased.
Let's denote them x and y.

So our purchase will have a weight of $10x + 5y$ grams, and a price of $5x + 7y$ cents.Thus we'll want a solution to the system of equations:

$$\begin{cases} 10x + 5y = 1000 \longleftarrow \text{weight of the candy in grams} \\ 5x + 7y = \ \ 800 \longleftarrow \text{cost of the candy in cents} \end{cases}$$

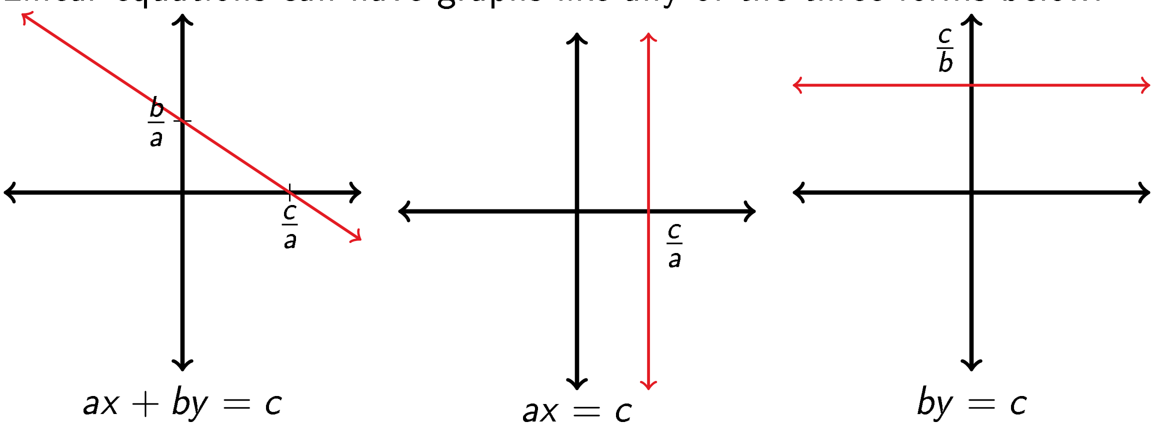
How to Solve!

Once we have systems of equations, we want to find their *solutions*. There are several approaches to doing so.

- Graphing
- Algebraic substitution
- Algebraic elimination
- Matrix methods

Graphing a Linear Equation

Recall that a *linear equation* is so called because its graph is a line. Linear equations can have graphs like any of the three forms below:



A system of two equations, then, can be represented by a pair of lines.

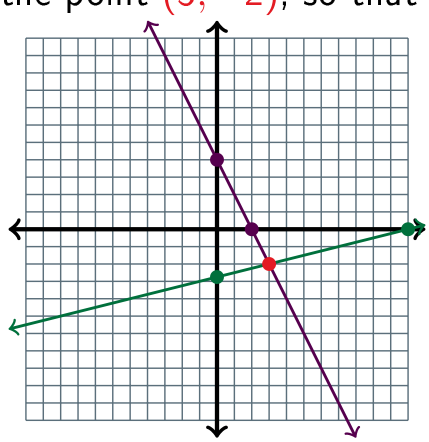
Graphing and Interpreting a System

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases}$$

The first line passes through $(11, 0)$ and $(0, \frac{-11}{4})$.

The second line passes through $(2, 0)$ and $(0, 4)$.

And they intersect at the point $(3, -2)$, so that's the solution!



Graphing in Context

- Advantages:
  - Highly visual
  - Easily interpreted
  - Can be done with almost no algebraic skill
- Disadvantages:
  - Requires tools
  - May be difficult to do exactly
  - Non-integer solutions are difficult to interpret

Single-Variable Solution

A linear equation in a *single* variable is easy to solve, for example:

$$\begin{aligned} 5x + 3 &= 19 \\ -3 &\quad -3 \\ 5x &= 11 \\ \div 5 &= \div 5 \\ x &= \frac{11}{5} = 2.2 \end{aligned}$$

We can use the same technique to solve for the variables in a system of equations one at a time.

The Substitution Method

Let's consider our old friend:

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases}$$

We can take *one* of these equations and solve for *one* variable; let's take the second equation and solve for x.

$$\begin{aligned} 2x + y &= 4 \\ -2x \qquad & \quad -2x \\ y &= 4 - 2x \end{aligned}$$

Now we can *substitute* this relationship back into the first equation:

$$\begin{aligned} x - 4y &= 11 \\ x - 4(4 - 2x) &= 11 \end{aligned}$$

The Substitution Method, continued

The equation at the end of the last slide is a linear equation in one variable.

$$\begin{aligned} x - 4(4 - 2x) &= 11 \\ x - 16 + 8x &= 11 \\ 9x - 16 &= 11 \\ 9x &= 27 \\ x &= 3 \end{aligned}$$

And now,from the rule that $y = 4 - 2x$, we can find that $y = 4 - 2 \cdot 3 = -2$.

The General Approach

To perform substitution, go throguh the following steps:

1. Choose one of the two equations, and solve it for one of the variables.
2. Substitute the variable you solved for into the second equation.
3. Solve the second equation, which is now in terms of a single variable.
4. Substitute the solution you find back into your solution from step 1.

In step one you seem to be making an arbitrary decision. But whichever choice you make will work.

The Same Story, Told Twice

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases}$$

$2x + y = 4$	$x - 4y = 11$
$y = 4 - 2x$	$y = \frac{x-11}{4}$
$x - 4y = 11$	$2x + y = 4$
$x - 4(4 - 2x) = 11$	$2x + \frac{x-11}{4} = 4$
$x - 16 + 8x = 11$	$2x + \frac{1}{4}x - \frac{11}{4} = 4$
$9x - 16 = 11$	$\frac{9}{4}x - \frac{11}{4} = 4$
$9x = 27$	$\frac{9}{4}x = \frac{27}{4}$
$x = 3$	$x = 3$
$y = 4 - 2x$	$y = \frac{x-11}{4}$
$y = 4 - 2 \cdot 3 = -2$	$y = \frac{3-11}{4} = -2$

Equation-Manipulation Techniques

In a system of two equations, the following operations don't change the solution of the system:

- Swapping the two equations
- Multiplying one equation by a nonzero number
- Adding one equation to the other

In addition, using the second and third operation, we can also:

- Add a multiple of one equation to another

We can *eliminate* a variable from one equation with these operations.

An elimination example

Let's return to our previously solved problem:

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases}$$

We can multiply the first equation by -2 to get two equations which, when added together, will have no x:

$$\begin{cases} -2x + 8y = -22 \\ 2x + y = 4 \end{cases}$$

And now we can add the second equation to the first:

$$\begin{cases} 9y = -18 \\ 2x + y = 4 \end{cases}$$

An elimination example, continued

$$\begin{cases} \quad\;\; 9y = -18 \\ 2x + \;\; y = \;\; 4 \end{cases}$$

Now, we can multiply the first equation by $\frac{1}{9}$:

$$\begin{cases} \qquad y = -2 \\ 2x + y = \;\; 4 \end{cases}$$

And subtract it from the second equation:

$$\begin{cases} \qquad y = -2 \\ 2x \qquad = \;\; 6 \end{cases}$$

Finally, we divide the second equation by 2:

$$\begin{cases} \qquad y = -2 \\ x \qquad = \;\; 3 \end{cases}$$

Contradictory systems

Sometimes, these methods *don't* produce a solution:

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = 5 \end{cases}$$

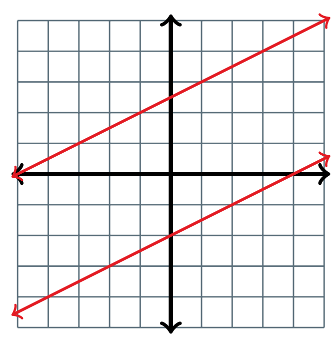
$$\begin{aligned} -x + 2y &= 5 \\ y &= \tfrac{x+5}{2} \end{aligned}$$

$$\begin{aligned} 2x - 4y &= 8 \\ 2x - 4\tfrac{x+5}{2} &= 8 \\ 2x - 2x - 10 &= 8 \\ -10 &= 8?? \end{aligned}$$

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = 5 \end{cases}$$

$$\begin{cases} x - 2y = 4 \\ -x + 2y = 5 \end{cases}$$

$$\begin{cases} x - 2y = 4 \\ \qquad 0 = 9?? \end{cases}$$



With both approaches, we arrive at a *contradiction*, so this equation has *no* solution.

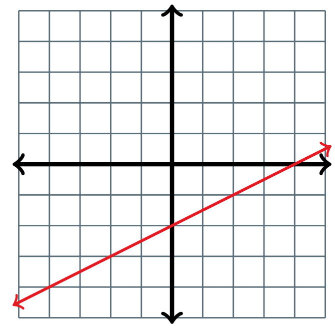
Underspecified systems

Sometimes there's no contradiction, but not a unique solution either:

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = -4 \end{cases}$$

$$-x + 2y = -4$$
$$y = \frac{x-4}{2}$$
$$2x - 4y = 8$$
$$2x - 4\frac{x-4}{2} = 8$$
$$2x - 2x + 8 = 8$$
$$8 = 8??$$

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = -4 \end{cases}$$
$$\begin{cases} x - 2y = 4 \\ -x + 2y = -4 \end{cases}$$
$$\begin{cases} x - 2y = 4 \\ 0 = 0?? \end{cases}$$



With both approaches, we arrive at a *tautology*, so this equation has *infinitely many* solutions.

Summary of Techniques and Interpretation

We have seen three approaches to interpreting and solving systems:

- Graphing — not recommended for finding solutions!
- Substitution
- Elimination

One of three results can arise:

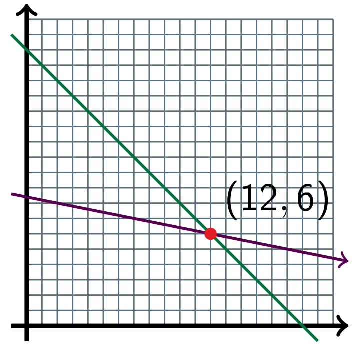
- Completed process or two intersecting lines (one solution)
- Contradictory result or two parallel lines (no solution)
- Tautological result or two coincident lines (many solutions)

Application #1 revisited

$$\begin{cases} x + y = 18 \longleftarrow \text{number of coins} \\ x + 5y = 42 \longleftarrow \text{value of coins} \end{cases}$$

$$\begin{aligned} x + y &= 18 \\ y &= 18 - x \\ \hline x + 5y &= 42 \\ x + 5(18 - x) &= 42 \\ x + 90 - 5x &= 42 \\ -4x &= -48 \\ x &= 12 \\ \hline y &= 18 - 12 \\ y &= 6 \end{aligned}$$

$$\begin{cases} x + y = 18 \\ x + 5y = 42 \end{cases} \quad \begin{cases} x + y = -18 \\ 4y = 24 \end{cases} \quad \begin{cases} -x - y = -18 \\ y = 6 \end{cases} \quad \begin{cases} -x = -12 \\ y = 6 \end{cases}$$



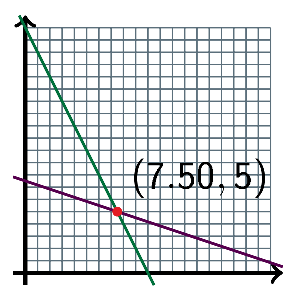
Thus, we have 12 pennies and 6 nickels.

Application #2 revisited

$$\begin{cases} 2x + y = 20 \quad\quad \longleftarrow \text{cost of 2 adults and a child} \\ x + 3y = 22.50 \longleftarrow \text{cost of an adult and 3 children} \end{cases}$$

$$\begin{aligned} 2x + y &= 20 \\ y &= 20 - 2x \\ \hline x + 3y &= 22.5 \\ x + 3(20 - 2x) &= 22.5 \\ x + 60 - 6x &= 22.5 \\ -5x &= -37.5 \\ x &= 7.50 \\ \hline y &= 20 - 2 \cdot 7.5 \\ y &= 5 \end{aligned}$$

$$\begin{cases} 2x + y = 20 \\ 2x + 6y = 45 \end{cases} \quad \begin{cases} 2x + y = 20 \\ 5y = 25 \end{cases} \quad \begin{cases} 2x + y = 20 \\ y = 5 \end{cases} \quad \begin{cases} 2x = 15 \\ y = 5 \end{cases}$$



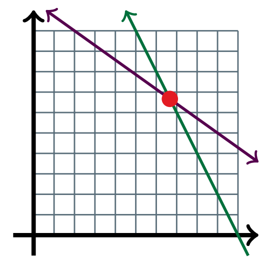
Thus, the price will be \$7.50 for an adult and \$5.00 for a child.

Application #3 revisited

$$\begin{cases} 10x + 5y = 1000 \longleftarrow \text{weight of the candy in grams} \\ 5x + 7y = \ \ 800 \longleftarrow \text{cost of the candy in cents} \end{cases}$$

$$\begin{aligned} 10x + 5y &= 1000 \\ y &= 200 - 2x \\ \hline 5x + 7y &= 800 \\ 5x + 7(200 - 2x) &= 800 \\ 5x + 1400 - 14x &= 800 \\ -9x &= 600 \\ x &= \tfrac{200}{3} \approx 67 \\ \hline y &= \tfrac{200}{3} \approx 67 \end{aligned}$$

$$\begin{cases} 10x + \ \ 5y = 1000 \\ 10x + 14y = 1600 \end{cases} \quad \begin{cases} 10x + 5y = 1000 \\ 9y = \ \ 600 \end{cases} \quad \begin{cases} 10x + 5y = 1000 \\ y = \frac{200}{3} \end{cases} \quad \begin{cases} 10x = \frac{2000}{3} \\ y = \frac{200}{3} \end{cases}$$



Thus, we want about 67 chocolates and 67 caramels.