Systems of Linear Equations and Matrices

MATH 107: Finite Mathematics

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Matrices

A *matrix* is a rectangle of numbers enclosed by square brackets.

A matrix is of *size* $m \times n$ if it has m rows and n columns.

The rows and columns of a matrix are labeled in order starting at 1.

An $m \times 1$ matrix is called a *column matrix or column vector*.
A $1 \times n$ matrix is called a *row matrix or row vector*.
An $n \times n$ matrix is called a *square matrix of order n*.

Col. 1 Col. 3
| Col. 2 | Col. 4
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$$3 \times 4 \text{ matrix: } \begin{bmatrix} 3 & 5 & 8 & -2 \\ 8 & 1 & -1 & 0 \\ -6 & 5 & 5 & 0 \end{bmatrix} \begin{matrix} \longleftarrow \text{Row 1} \\ \longleftarrow \text{Row 2} \\ \longleftarrow \text{Row 3} \end{matrix}$$

Parts of a matrix

A matrix can be named (with a capital letter).

To refer to a single *element* of a matrix A, we use the subscripted form a_{ij}.

The *principal diagonal* of a matrix A is the sequence $a_{11}, a_{22}, \ldots$.

Example

$$A = \begin{bmatrix} 3 & 5 & \mathbf{8} & -2 \\ 8 & 1 & -1 & \mathbf{0} \\ -6 & 5 & 5 & 0 \end{bmatrix}$$

$a_{13} = 8$; $a_{24} = 0$; a_{44} doesn't exist.
The principal diagonal of A is the sequence $3, 1, 5$.

Matrices and systems of equations

A system of equations can be encoded into matrices, with the relevant information being the *coefficients* on the left side of the system and *constants* on the right side.

Example

System of equations: $\begin{cases} 2x_1 + 5x_2 = 8 \\ x_1 - 4x_2 = 0 \end{cases}$

Coefficient matrix: $\begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix}$

Constant vector: $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$

More general systems

Matrix presentation is a concise way to present a system of any number of equations in any number of variables:

Example

Consider the system:

$$\begin{cases} 2x_1 + 3x_2 \phantom{{}+6x_3} = 2 \\ 4x_1 - x_2 + 6x_3 = -13 \\ 5x_1 + 2x_2 + x_3 = 0 \\ \phantom{5x_1 + {}} 2x_2 - 3x_3 = 10 \end{cases}$$

This has coefficient matrix $\begin{bmatrix} 2 & 3 & 0 \\ 4 & -1 & 6 \\ 5 & 2 & 1 \\ 0 & 2 & -3 \end{bmatrix}$ and constant vector $\begin{bmatrix} 2 \\ -13 \\ 0 \\ 10 \end{bmatrix}$.

In general, a system of m equations in n variables has a $m \times n$ coefficient matrix, and a $m \times 1$ constant vector.

Augmented matrices

It's easier to handle a single matrix than a pair, so we often squash these two together to form an *augmented matrix*:

$$\begin{cases} 2x_1 + 5x_2 = 8 \\ x_1 - 4x_2 = 0 \end{cases} \longrightarrow \begin{bmatrix} 2 & 5 \\ 1 & -4 \end{bmatrix} \;\&\; \begin{bmatrix} 8 \\ 0 \end{bmatrix} \longrightarrow \left[\begin{array}{cc|c} 2 & 5 & 8 \\ 1 & -4 & 0 \end{array}\right]$$

In general, a system of m equations in n variables is associated with an $m \times (n+1)$ augmented matrix.

Modifying matrices

When we learned the *elimination method*, we saw three ways to modify a system without changing its solutions:

- Swap the positions of two equations.
- Multiply an entire equation by a nonzero number.
- Add a multiple of an equation to another equation.

Since the *rows* of an augmented matrix correspond to equations, we could correspondingly modify matrices by:

- swapping the positions of two rows.
- multiplying an entire row by a nonzero number.
- adding a multiple of a row to another row.

Examples of row operations

Let's consider a matrix $A = \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & 1 & 0 & -4 \\ 6 & 8 & -10 & 0 \end{bmatrix}$, and explore how it is changed by the row-operations:

$$A \overset{R_1 \leftrightarrow R_3}{\sim} \begin{bmatrix} 6 & 8 & -10 & 0 \\ 0 & 1 & 0 & -4 \\ 2 & 5 & -3 & 1 \end{bmatrix}$$

$$A \overset{R_2 - 3R_1 \to R_2}{\sim} \begin{bmatrix} 2 & 5 & -3 & 1 \\ -6 & -14 & 9 & -7 \\ 6 & 8 & -10 & 0 \end{bmatrix}$$

$$A \overset{R_3 \div 2 \to R_3}{\sim} \begin{bmatrix} 2 & 5 & -3 & 1 \\ 0 & 1 & 0 & 4 \\ 3 & 4 & -5 & 0 \end{bmatrix}$$

If one matrix can be obtained from another through row operations, we call them *row-equivalent*.

Solving systems with row-operations

Using matrices, we can perform elimination methods much more compactly than before:

$$\begin{cases} x - 4y = 11 \\ 2x + y = 4 \end{cases} \Longrightarrow \left[\begin{array}{cc|c} 1 & -4 & 11 \\ 2 & 1 & 4 \end{array}\right] \underset{\sim}{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 9 & -18 \end{array}\right] \underset{\sim}{R_2 \div 9 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & -4 & 11 \\ 0 & 1 & -2 \end{array}\right] \underset{\sim}{R_1 + 4R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right] \Longrightarrow \begin{cases} x = 3 \\ y = -2 \end{cases}$$

This, as we shall see, is both the quickest and most systematizable way of solving a system of equations.

Larger systems

Row operations can make short work of *larger* systems, too.

$$\begin{cases} 3x + 7y - z = 11 \\ x + 2y - z = 3 \\ 2x + 4y - 2z = 10 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 3 & 7 & -1 & 11 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & -2 & 10 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & 7 & -1 & 11 \\ 2 & 4 & -2 & 10 \end{array}\right]$$

$$\underset{\sim}{R_2 - 3R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -4 & 2 \\ 2 & 4 & -2 & 10 \end{array}\right] \underset{\sim}{R_3 - 2R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & -4 & 4 \end{array}\right] \underset{\sim}{R_1 - 2R_2 \to R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & -4 & 4 \end{array}\right] \underset{\sim}{R_3 \div (-4) \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 9 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right] \underset{\sim}{R_2 + 4R_3 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 9 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \underset{\sim}{R_1 - 5R_3 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \Longrightarrow \begin{cases} x = 8 \\ y = -2 \\ z = -1 \end{cases}$$

Unusual systems

Recall that sometimes systems of equations have *no* solution, or *infinitely many* solutions. Let's look at the associated matrices:

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = -4 \end{cases} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 8 \\ -1 & 2 & -4 \end{array}\right] \underset{\sim}{R_1 \div 2 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ -1 & 2 & -4 \end{array}\right]$$

$$\underset{\sim}{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

so an *underspecified* system will have a row $\left[\begin{array}{cc|c} 0 & 0 & 0 \end{array}\right]$.

$$\begin{cases} 2x - 4y = 8 \\ -x + 2y = 5 \end{cases} \Longrightarrow \left[\begin{array}{cc|c} 2 & -4 & 8 \\ -1 & 2 & 5 \end{array}\right] \underset{\sim}{R_1 \div 2 \to R_1} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ -1 & 2 & 5 \end{array}\right]$$

$$\underset{\sim}{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -2 & 4 \\ 0 & 0 & 9 \end{array}\right]$$

so an *overspecified* system will have a row $\left[\begin{array}{cc|c} 0 & 0 & k \end{array}\right]$, with $k \neq 0$.

Summary

In general, we can convert a system to an augmented matrix:

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \Longrightarrow \left[\begin{array}{cc|c} a & b & c \\ d & e & f \end{array}\right]$$

We then use the *row operations* to build a *row-equivalent* matrix of one of the following three forms:

Matrix:	$\left[\begin{array}{cc\|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$	$\left[\begin{array}{cc\|c} 1 & m & n \\ 0 & 0 & k \end{array}\right]$ $(k \neq 0)$
Number of solutions:	One	Infinitely many	Zero
Description of solutions:	$x = m,\ y = n$	$x = n - my$	