Row-reduction and Gauss-Jordan Elimination

MATH 107: Finite Mathematics

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Desired matrix forms

We saw previously that every 2×3 augmented matrix is row-equivalent to one of the three matrices:

$$\left[\begin{array}{cc|c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & m & n \\ 0 & 0 & k \end{array}\right]$$

Larger matrices are row-equivalent to similar structures. Such matrices are said to be in *reduced row-echelon form*.

Definition of reduced form

$$\begin{bmatrix}1 & 0 & m\\ 0 & 1 & n\end{bmatrix} \quad \begin{bmatrix}1 & m & n\\ 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & m & n\\ 0 & 0 & 1\end{bmatrix}$$

Definition of reduced form

A matrix is in *reduced row-echelon form* (or *reduced form*) if:

- the all-zero rows are at the bottom of the matrix,
- the leftmost nonzero element in each row (called a *pivot*) is 1,
- the pivot of each row is to the right of the pivots in higher rows, and
- every other element of a column containing a pivot is zero,

Examples of reduced form

As a practice, we can ask if each of the following matrices is row-reduced:

- $\begin{bmatrix}0 & 1 & -3 & 0 & 2\\ 0 & 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$ is row-reduced; note the pivots.
- $\begin{bmatrix}1 & 4 & 0 & 3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ is not row-reduced; nonzero entry in pivot column.
- $\begin{bmatrix}0 & 6 & -5\\ 1 & 0 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ is not row-reduced, because a pivot is 6, and it is above and to the right of another pivot.

Row-equivalency to reduced forms

Each of the problems on the previous slides can be "fixed" via row operations:

- $\begin{bmatrix} 1 & 4 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_1 - 3R_3 \to R_1} \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- $\begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 6 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{R_2 \div 6 \to R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{-5}{6} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

The procedure by which we convert any matrix to a reduced equivalent is called *Gauss-Jordan elimination*.

Underlying techniques of Gauss-Jordan Elimination

Each requirement of reduced form can be addressed with a row operation.

- To get rows of zero at the bottom, we can *swap* rows.
- To get each pivot to be 1, we can *scale* its row.
- To get pivots arranged from upper left to lower right, we can *swap* rows.
- To get entries in the same column as a pivot to zero, we can *add/subtract* the pivot row to them.

If we work from left to right, selecting as a pivot the top nonzero entry of each column which doesn't already have a pivot, this is actually a methodical operation!

Example of the underlying approach

Let's row-reduce a sample matrix.
Column 1 has a nonzero entry, marked as a pivot.
Column 2 has a nonzero entry in the second row; it's a pivot.

$$\begin{bmatrix} 0 & 5 & 0 & -8 \\ 3 & -6 & 0 & 12 \\ 0 & -15 & 0 & 24 \end{bmatrix} \sim \begin{bmatrix} 3 & -6 & 0 & 12 \\ 0 & 5 & 0 & -8 \\ 0 & -15 & 0 & 24 \end{bmatrix} \quad (R_2 \leftrightarrow R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 5 & 0 & -8 \\ 0 & -15 & 0 & 24 \end{bmatrix} \quad (R_1 \div 3 \to R_1)$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 1 & 0 & -\frac{8}{5} \\ 0 & -15 & 0 & 24 \end{bmatrix} \quad (R_2 \div 5 \to R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \frac{4}{5} \\ 0 & 1 & 0 & -\frac{8}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} R_1 + 2R_2 \to R_1 \\ R_3 + 15R_2 \to R_3 \end{pmatrix}$$

The Gauss-Jordan Process

Here's the process explicitly laid out:

1. Take the leftmost column which has neither a selected pivot nor all zero elements to be your *active column*. If there is no such column, we're done!
2. Let the uppermost row which has no pivot but does have a nonzero entry in this column be your *active row*, and let the element where they intersect be this row's *pivot*. If there is no such row, we're done!
3. If there are rows without pivots above the active row, swap it with them.
4. Scale the active row by dividing by the value of the pivot.
5. Subtract the active row times the entries above and below its pivot from the rows above and below it.
6. Go back to step 1.

Working through the process

1	2	3	4	5
Select column	Select row	Swap	Scale	Subtract

$$\begin{bmatrix} 0 & -2 & 8 & 1 \\ 2 & -2 & 6 & -4 \\ 0 & -1 & 4 & \frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 6 & -4 \\ 0 & -2 & 8 & 1 \\ 0 & -1 & 4 & \frac{1}{2} \end{bmatrix} \quad (R_2 \leftrightarrow R_1)$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & -2 & 8 & 1 \\ 0 & -1 & 4 & \frac{1}{2} \end{bmatrix} \quad (R_1 \div 2 \to R_1)$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 1 & -4 & -\frac{1}{2} \\ 0 & -1 & 4 & \frac{1}{2} \end{bmatrix} \quad (R_2 \div (-2) \to R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -\frac{5}{2} \\ 0 & 1 & -4 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} R_1 + R_2 \to R_1 \\ R_3 + R_2 \to R_3 \end{pmatrix}$$

Returning to equations

So the Gauss-Jordan elimination method eventually gets you a solution to a system of equations:

$$\begin{cases} -2y + 8z = 1 \\ 2x - 2y + 6z = -4 \\ -y + 4z = \frac{1}{2} \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 0 & -2 & 8 & 1 \\ 2 & -2 & 6 & -4 \\ 0 & -1 & 4 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{5}{2} \\ 0 & 1 & -4 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\begin{cases} x - z = -\frac{5}{2} \\ y - 4z = -\frac{1}{2} \\ 0 = 0 \end{cases}$$

so $x = z - \frac{5}{2}$, and $y = 4z - \frac{1}{2}$, but z could be anything; there are infinitely many solutions.