Matrix Arithmetic

MATH 107: Finite Mathematics

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Why matrices?

Up until now, we've used matrices as a convenient way to represent systems of equations: row operations are just a way of "encoding" transformations of systems.

However, matrices can be processed in some ways which *aren't* like what we've already seen with systems of equations!

Here we will treat matrices like *algebraic objects*, describing arithmetic operations on them. The study of matrices in this context is called *linear algebra*.

A step backwards

When building an arithmetic for matrices, we might ask: how does our ordinary arithmetic work?

Arithmetic is described chiefly by addition and multiplication, and we can characterize arithmetic with a number of *laws*:

Commutative Laws $a + b = b + a$ and $ab = ba$.

Associative Laws $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

Identity Laws $a + 0 = a$ and $a \cdot 1 = a$.

Inverse Laws $a + (-a) = 0$ and *if* $a \neq 0$, then $a \cdot \frac{1}{a} = 1$.

Distributive Law $a(b + c) = ab + ac$.

Note that the inverse laws additionally give us a concept of "subtraction" and "division".

As we build an arithmetic for matrices, we will keep these laws in mind.

How We Add Matrices

Matrices of the same size are added *elementwise*, and we just add them term by term.

Example

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 3 \\ 5 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 4 + (-2) & -1 + 1 & 0 + 3 \\ 2 + 5 & 1 + 6 & 3 + (-8) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 7 & 7 & -5 \end{bmatrix}.$$

Matrices of *different* sizes simply can't be added.

Non-example

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 3 & 5 \\ 6 & -8 \end{bmatrix}$$ doesn't exist.

Ramifications of Matrix Addition

Since addition is term-by-term, the laws which were true about adding numbers are true about matrices too. If A, B, and C are all $m \times n$ matrices, then they obey:

Commutative Law of + $A + B = B + A$.

Associative Law of + $(A + B) + C = A + (B + C)$.

Identity Law of + $A + \mathbf{0}_{m\times n} = A$.

Inverse Law of + $A + (-A) = \mathbf{0}_{m\times n}$.

Above, $\mathbf{0}_{m\times n}$ is an $m \times n$ matrix whose elements are all zero; $(-A)$ is an $m \times n$ matrix whose elements are the negations of the elements of A.

Among other consequences, we can now use $A - B$ as a shorthand for $A + (-B)$.

Matrix Subtraction Explored

You can think of subtraction either as a variant on matrix addition, or as just elementwise subtraction.

Example

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 3 \\ 5 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -1 & -3 \\ -5 & -6 & 8 \end{bmatrix} =$$

$$\begin{bmatrix} 4+2 & -1+(-1) & 0+(-3) \\ 2+(-5) & 1+(-6) & 3+8 \end{bmatrix} = \begin{bmatrix} 6 & -2 & -3 \\ -3 & -5 & 11 \end{bmatrix} \text{ or:}$$

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} -2 & 1 & 3 \\ 5 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 4-(-2) & -1-1 & 0-3 \\ 2-5 & 1-6 & 3-(-8) \end{bmatrix} =$$

$$\begin{bmatrix} 6 & -2 & -3 \\ -3 & -5 & 11 \end{bmatrix}$$

As with addition, you can't subtract matrices of different sizes.

A simple form of multiplication

It would make a lot of sense to call $A + A$ by the name "$2A$", or let $A + A + A = 3A$. For this reason, we have the concept of *scalar multiplication* (a *scalar* is a number, in contrast to a matrix).

For any matrix A and number k, we denote by kA the result of multiplying k by every element of A.

Example

$$5\begin{bmatrix} 3 & 4 & -2 \\ -1 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 5\cdot 3 & 5\cdot 4 & 5(-2) \\ 5(-1) & 5\cdot 0 & 5\cdot 12 \end{bmatrix} = \begin{bmatrix} 15 & 20 & -10 \\ -5 & 0 & 60 \end{bmatrix}.$$

$$\frac{-1}{3}\begin{bmatrix} 3 & 4 & -2 \\ -1 & 0 & 12 \end{bmatrix} = \begin{bmatrix} -1 & \frac{-4}{3} & \frac{-2}{3} \\ \frac{1}{3} & 0 & -4 \end{bmatrix}$$

Conventionally this multiplication is always written kA, not Ak. Also, $(\frac{1}{k})A$ is not generally written $\frac{A}{k}$.

Notable properties of scalar multiplication

Scalar multiplication is "well-behaved" in a lot of ways; as previously, here we assume A is an $m \times n$ matrix.

- **Associative Law of Scalar ×** $k(\ell A) = (k\ell)A$.
- **Identity Law of Scalar ×** $1A = A$.
- **Zero Law of Scalar ×** $0A = \mathbf{0}_{m\times n}$.
- **Scalar Distributive Laws** $k(A + B) = kA + kB$ and $(k + \ell)A = kA + \ell A$.

Making matrices fruitful

There is an obvious way to multiply matrices which is, unfortunately, *not correct*:

Elementwise matrix multiplication

$$\begin{bmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 5 & 6 & -8 \end{bmatrix} \neq \begin{bmatrix} 4(-2) & -1 \cdot 1 & 0 \cdot 3 \\ 2 \cdot 5 & 1 \cdot 6 & 3(-8) \end{bmatrix}.$$

The actual matrix multiplication procedure is less obvious, but turns out to be more useful.

Row-and-column products

The underlying concept of matrix multiplication is the idea of multiplying a *row of n numbers* by a *column of n numbers*, as such:

Row-and-column multiplication process

To multiply $3 \quad -1 \quad 8 \quad 0$ by $\begin{matrix} 7 \\ 3 \\ -2, \\ 12 \end{matrix}$ we first multiply each pair of terms, and then add the products.

$$3 \cdot 7 + (-1)3 + 8(-2) + 0 \cdot 12 = 21 - 3 - 16 + 0 = 2$$

Many row-and-column products

And multiplying two matrices is a matter of multiplying *every* row of the first matrix by *every* column of the second.

Example

$$\begin{bmatrix} 7 & 2 & -1 \\ 3 & 8 & 0 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 0 & 3 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 7(-2)+2\cdot 0+(-1)1 & 7\cdot 4+2\cdot 3+(-1)0 \\ 3(-2)+8\cdot 0+0\cdot 1 & 3\cdot 4+8\cdot 3+0\cdot 0 \end{bmatrix} = \begin{bmatrix} -15 & 34 \\ -6 & 36 \end{bmatrix}$$

Notice, curiously, that the product of a 2×3 matrix and a 3×2 matrix is a 2×2 matrix!

Size restrictions

To multiply AB, the rows of A and the columns of B must be the same size.

If A is $m \times n$, its rows have n terms.
If B is $p \times q$, its columns have p terms.

So AB only exists if $n = p$! And AB itself will have size $m \times q$.

Note that A^2 (a shorthand for AA) only exists when A is a square matrix.

How matrix multiplication is perverse

The results we just saw demonstrate that matrix multiplication is *noncommutative* — AB and BA could be different matrices, different sizes, or even different states of existence!

Noncommutativity examples

$\begin{bmatrix}1 & 4\\0 & 2\end{bmatrix}\begin{bmatrix}0 & 1\\5 & 1\end{bmatrix} = \begin{bmatrix}20 & 5\\10 & 2\end{bmatrix}$, but $\begin{bmatrix}0 & 1\\5 & 1\end{bmatrix}\begin{bmatrix}1 & 4\\0 & 2\end{bmatrix} = \begin{bmatrix}0 & 2\\5 & 22\end{bmatrix}$.

$\left[\begin{smallmatrix}7 & 2 & 1\\3 & 8 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}2 & 4\\0 & 3\\1 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}15 & 34\\6 & 36\end{smallmatrix}\right]$, but $\left[\begin{smallmatrix}2 & 4\\0 & 3\\1 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}7 & 2 & 1\\3 & 8 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}26 & 36 & 2\\9 & 24 & 0\\7 & 2 & 1\end{smallmatrix}\right]$.

$\left[\begin{smallmatrix}2 & 4\\0 & 3\end{smallmatrix}\right]\left[\begin{smallmatrix}7 & 2 & 1\\3 & 8 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}26 & 36 & 2\\9 & 24 & 0\end{smallmatrix}\right]$, but $\left[\begin{smallmatrix}7 & 2 & 1\\3 & 8 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}2 & 4\\0 & 3\end{smallmatrix}\right]$ doesn't exist.

Thus, you have to be very careful about the order of multiplication!

How matrix multiplication is nice

Although matrix multiplication behaves badly in some ways, in others it is exactly what we expect; below, A, B, and C are matrices, and we assume A is of size $m \times n$:

Noncommutativity of Matrix $\times$	AB and BA may be unequal.
Commutative Law of Matrix/Scalar $\times$	$k(AB) = (kA)B = A(kB)$.
Associative Law of Matrix $\times$	$(AB)C = A(BC)$.
Identity Law of Matrix $\times$	$I_m A = A I_n = A$.
Distributive Laws of Matrix Arithmetic	$A(B+C) = AB + AC$ and $(A+B)C = AC + BC$.

Above, I_n is the $n \times n$ matrix $\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\0 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & \cdots & 0\\\vdots & \vdots & \vdots & \ddots & \vdots\\0 & 0 & 0 & \cdots & 1\end{bmatrix}$.

Why multiply like this?

Matrix multiplication has several uses, some of which are geometric, and some of which are practical. Suppose we have a concept of labor quantities and costs for different products and factories.

Production-pricing problem

Suppose we are making boats of 3 types, requring different labors. A canoe requires 2 hours of carpentry and an hour of welding; a sailboat needs 3 hours of carpentry, half an hour of welding, and 2 hours of sailmaking; a dinghy is made with an hour of carpentry and 2 hours of welding.

In Louisville, we pay carpenters \$18 per hour, welders \$15, and sailmakers \$20; in Lexington they pay carpenters \$20, welders \$14, and sailmakers \$16.

How much does it cost to make a boat of each type in each city?

We can represent this calculation with a *matrix product*.

Labor costs, cont'd

We can build a matrix whose rows represent the three types of boats, and whose columns are the types of labor; entries represent time:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & \frac{1}{2} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

and a matrix whose rows are the types of labor and whose columns are the cities; entries represent per-hour cost:

$$B = \begin{bmatrix} 18 & 20 \\ 15 & 14 \\ 20 & 16 \end{bmatrix}$$

and then our labor costs for each type of boat can be computed as a product:

$$AB = \begin{bmatrix} 51 & 54 \\ 101.50 & 99 \\ 48 & 48 \end{bmatrix}$$