Matrix Inverses

MATH 107: Finite Mathematics

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## What is an inverse?

Recall that in the context of ordinary numbers, the *multiplicative identity* is 1, and the *multiplicative inverse* of a number is something we multiply by it to get 1.

### Examples of multiplicative inverses

- $5 \cdot \frac{1}{5} = 1$.
- $\frac{-3}{8} \cdot \frac{-8}{3} = 1$.
- $(\pi + \sqrt{3}) \cdot \frac{1}{\pi+\sqrt{3}} = 1$.
- $0 \cdot x \neq 1$, no matter what $x$ is!

In general, *for any number $x$ except zero*, $x$ has a multiplicative inverse $\frac{1}{x}$.

Armed with that notion, we can ask: do matrices *also* have multiplicative inverses?

## Translating numbers to matrices

The equivalent concept to "1" in matrices is the *identity matrix*

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

So we might say that a matrix $B$ is an *inverse* of a matrix $A$ if both $AB$ and $BA$ are identity matrices.

**Example of matrix inverses**

$$\begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix} \begin{bmatrix} -13 & 8 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \text{ also, } \begin{bmatrix} -13 & 8 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So the idea we describe above definitely exists sometimes!





## Good news for modern matrices

There are a few very useful facts that make working with matrix inverses easier.

- If $A$ isn't square, there is *no* matrix $B$ such that $AB$ and $BA$ are both identity matrices.
- If $A$ is square, and $AB = I_n$, then $BA$ is also equal to $I_n$.
- If $A$ is square, there is *at most one* matrix $B$ such that $AB = I_n$, and $B$ is the same size as $A$.

As a result of these three facts, we can simplify our search:

- We won't consider nonsquare matrices at all.
- For a matrix $A$, we can be certain that its *inverse* $A^{-1}$, if it exists, is unique.
- We only need to perform one multiplication to test inverses.

# The cloud behind the silver lining

Unfortunately, not all square matrices have inverses!

## An uninvertable matrix

For *any* $2 \times 2$ matrix $B$, $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

This isn't a surprise, since the number 0 doesn't have an inverse either.

## A subtler problem

For *any* $3 \times 3$ matrix $B$, the product $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} B$ always has equal first and second rows, so $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} B \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Clearly, rather complicated rules govern which matrices have inverses.





# What we want to know

We now have two important questions to be asked about any square matrix $A$:

- Does $A^{-1}$ exist?
- If so, how do we compute it?

We can begin to answer the first question from what we've already seen.

## Simple danger signs

We know immediately that a matrix has *no inverse* if:

- Any row or column contains only zeroes, or
- any row or column is identical to another row or column.

These rules are incomplete, and the safest way to check if a matrix has an inverse is to try to find it.

# Calculating a matrix inverse

How might we find, say, $\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1}$?

Let's *assume* it has an inverse, and try to figure out what it would be.

$$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2a+4c & 2b+4d \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so in order for all the terms to be equal, we have to have all four of the following statements be true:

$$\begin{cases} 2a+4c=1 \\ -3a+4c=0 \end{cases}$$

$$\begin{cases} 2b+4d=0 \\ -3b+4d=1 \end{cases}$$





# Calculating a matrix inverse (continued)

Thus, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1}$ only if $a$, $b$, $c$, and $d$ are solutions to:

$$\begin{cases} 2a+4c=1 \\ -3a+4c=0 \end{cases} \Longrightarrow \begin{cases} a = \frac{1}{5} \\ c = \frac{3}{20} \end{cases}$$

and

$$\begin{cases} 2b+4d=0 \\ -3b+4d=1 \end{cases} \Longrightarrow \begin{cases} b = \frac{-1}{5} \\ d = \frac{1}{10} \end{cases}$$

But we know how to do these!

Thus: $\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{-1}{5} \\ \frac{3}{20} & \frac{1}{10} \end{bmatrix}$

# Failing to calculate a matrix inverse

How would this process work with a matrix that has no inverse? Let's try to find $\begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}^{-1}$.

$$\begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2a - 4c & 2b - 4d \\ -5a + 10c & -5b + 10d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} 2a - 4c = 1 \\ -5a + 10c = 0 \end{cases}$$
$$\begin{cases} 2b - 4d = 0 \\ -5b + 10d = 1 \end{cases}$$

These two systems have no solution!





# Simplifying our process

We might skip the named variables in the previous processes and jump directly to the matrix interpretation of a system of equations:

$$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} \begin{matrix} \nearrow \left[\begin{array}{cc|c} 2 & 4 & 1 \\ -3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0.2 \\ 0 & 1 & 0.15 \end{array}\right] \searrow \\ \searrow \left[\begin{array}{cc|c} 2 & 4 & 0 \\ -3 & 4 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -0.2 \\ 0 & 1 & 0.1 \end{array}\right] \nearrow \end{matrix} \begin{bmatrix} 0.2 & -0.2 \\ 0.15 & 0.1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}^{-1} \begin{matrix} \Rightarrow \left[\begin{array}{cc|c} 2 & -4 & 1 \\ -5 & 10 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0.5 \\ 0 & 0 & 2.5 \end{array}\right] \searrow \\ \Rightarrow \left[\begin{array}{cc|c} 2 & -4 & 0 \\ -5 & 10 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 1 \end{array}\right] \nearrow \end{matrix} \text{Failure!}$$

# Simplifying even further

Since we're reducing two augmented matrices with the same left side, augment one matrix with *two* extra columns so we don't repeat ourselves!

$$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix}^{-1} : \begin{matrix} \left[\begin{array}{cc|c} 2 & 4 & 1 \\ -3 & 4 & 0 \end{array}\right] \searrow \\ \left[\begin{array}{cc|c} 2 & 4 & 0 \\ -3 & 4 & 1 \end{array}\right] \nearrow \end{matrix} \left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ -3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0.2 & -0.2 \\ 0 & 1 & 0.15 & 0.1 \end{array}\right]$$

$$\begin{bmatrix} 2 & -4 \\ -5 & 10 \end{bmatrix}^{-1} : \begin{matrix} \left[\begin{array}{cc|c} 2 & -4 & 1 \\ -5 & 10 & 0 \end{array}\right] \searrow \\ \left[\begin{array}{cc|c} 2 & -4 & 0 \\ -5 & 10 & 1 \end{array}\right] \nearrow \end{matrix} \left[\begin{array}{cc|cc} 2 & -4 & 1 & 0 \\ -5 & 10 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -2 & 0.5 & 0 \\ 0 & 0 & 2.5 & 1 \end{array}\right]$$





# Take inverses with this one weird trick!

In the end, the procedure for inverting a matrix is a small variation on things we've seen before.

**Putting it all together**

Given an $n \times n$ matrix $A$:

1. Build the augmented matrix $\left[A \mid I_n\right]$.
2. Use Gauss-Jordan elimination to transform it into some $\left[A' \mid B\right]$.
3. If $A' = I_n$, then $B = A^{-1}$.
4. If $A' \neq I_n$, then $A$ has no inverse.

We'll work through a $3 \times 3$ example; it's long but not difficult.

# A $3 \times 3$ inverse

We will calculate $\begin{bmatrix} 0 & 4 & -2 \\ 2 & -14 & 4 \\ -4 & 8 & 1 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{rrr|rrr} 0 & 4 & -2 & 1 & 0 & 0 \\ 2 & -14 & 4 & 0 & 1 & 0 \\ -4 & 8 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 2 & -14 & 4 & 0 & 1 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ -4 & 8 & 1 & 0 & 0 & 1 \end{array}\right] \quad (R_1 \leftrightarrow R_2)$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -7 & 2 & 0 & 0.5 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ -4 & 8 & 1 & 0 & 0 & 1 \end{array}\right] \quad \left(R_1 \times \tfrac{1}{2}\right)$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -7 & 2 & 0 & 0.5 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 0 & -20 & 9 & 0 & 2 & 1 \end{array}\right] \quad (R_3 + 4R_1)$$





# A $3 \times 3$ inverse (continued)

$$\sim \left[\begin{array}{rrr|rrr} 1 & -7 & 2 & 0 & 0.5 & 0 \\ 0 & 4 & -2 & 1 & 0 & 0 \\ 0 & -20 & 9 & 0 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & -7 & 2 & 0 & 0.5 & 0 \\ 0 & 1 & -0.5 & 0.25 & 0 & 0 \\ 0 & -20 & 9 & 0 & 2 & 1 \end{array}\right] \quad \left(R_2 \times \tfrac{1}{4}\right)$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & -1.5 & 1.75 & 0.5 & 0 \\ 0 & 1 & -0.5 & 0.25 & 0 & 0 \\ 0 & 0 & -1 & 5 & 2 & 1 \end{array}\right] \quad \begin{pmatrix} R_1 + 7R_2 \\ R_3 + 20R_2 \end{pmatrix}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & -1.5 & 1.75 & 0.5 & 0 \\ 0 & 1 & -0.5 & 0.25 & 0 & 0 \\ 0 & 0 & 1 & -5 & -2 & -1 \end{array}\right] \quad (R_3 \times (-1))$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -5.75 & -2.5 & -1.5 \\ 0 & 1 & 0 & -2.25 & -1 & -0.5 \\ 0 & 0 & 1 & -5 & -2 & -1 \end{array}\right] \quad \begin{pmatrix} R_1 + 1.5R_3 \\ R_2 + 0.5R_3 \end{pmatrix}$$