Algebra with Matrices

MATH 107: Finite Mathematics

University of Louisville

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Everyday algebra

Recall how we might solve an ordinary algebraic expression for x, as such:

$$\begin{aligned} 5x + 2 &= 16 \\ -2 \quad & -2 \\ 5x \quad &= 14 \\ \times\frac{1}{5} \quad & \times\frac{1}{5} \\ x &= 2.8 \end{aligned}$$

Except possibly for using "multiplication by $\frac{1}{5}$" instead of "division by 5", this should look very familiar.

Note that "multiplication by the inverse of 5" clears out that 5; thus, on an algebraic level, inverses are important.

From the mundane to the matrix

Let's consider a very similar-looking algebra problem to the last page's, but now everything (including x!) is a matrix:

$$\begin{aligned}
\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 4 \\ 12 \end{bmatrix} &= \begin{bmatrix} 0 \\ -3 \end{bmatrix} \\
-\begin{bmatrix} 4 \\ 12 \end{bmatrix} & \quad -\begin{bmatrix} 4 \\ 12 \end{bmatrix} \\
\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} X &= \begin{bmatrix} -4 \\ -15 \end{bmatrix} \\
\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \times & \quad \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \times \\
\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} X &= \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ -15 \end{bmatrix}
\end{aligned}$$

From the mundane to the matrix (continued)

$$\begin{aligned}
\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix} X &= \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ -15 \end{bmatrix} \\
X &= \begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 \\ -15 \end{bmatrix}
\end{aligned}$$

so if we knew $\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1}$, we could calculate X!

Note that $\begin{bmatrix} 3 & 8 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{-1}{13} & \frac{8}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix}$, so

$$X = \begin{bmatrix} \frac{-1}{13} & \frac{8}{13} \\ \frac{2}{13} & \frac{-3}{13} \end{bmatrix} \begin{bmatrix} -4 \\ -15 \end{bmatrix} = \begin{bmatrix} \frac{-116}{13} \\ \frac{37}{13} \end{bmatrix}$$

We thus have the means to *solve algebraic equations for vectors*!

We can solve algebra problems. So what?

Pleasingly enough, we can come full circle and bring our word problems and systems back to matrix algebra. Recall this word problem:

Quiz Question #1.1, or Text Question 4.1.70

A fruit grower uses two brands of fertilizer. Each bag of brand A contains 8 pounds of nitrogen and 4 pounds of phosphoric acid, while each bag of brand B contains 7 pounds of nitrogen and 6 pounds of phosphoric acid. Tests indicate that the grove needs 720 pounds of nitrogen and 500 pounds of phosphoric acid.

The solution to the question setup

Where x and y represent the respective number of bags of A and B, we wnat to satisfy

$$\begin{cases} 8x + 7y = 720 \\ 4x + 6y = 500 \end{cases}$$

Rephrasing systems as matrix equations

We could rewrite these two equations as an equality of vectors, and then factor the left side of the equation into a product of vectors.

$$\begin{cases} 8x + 7y = 720 \\ 4x + 6y = 500 \end{cases} \iff \begin{bmatrix} 8x + 7y \\ 4x + 6y \end{bmatrix} = \begin{bmatrix} 720 \\ 500 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 720 \\ 500 \end{bmatrix}$$

Now, since we'd really like to know what $\begin{bmatrix} x \\ y \end{bmatrix}$ is, we can solve the associated matrix algebra problem.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 720 \\ 500 \end{bmatrix} = \begin{bmatrix} 0.30 & -0.35 \\ -0.20 & 0.40 \end{bmatrix} \begin{bmatrix} 720 \\ 500 \end{bmatrix} = \begin{bmatrix} 41 \\ 56 \end{bmatrix}$$

So we end up needing 41 bags of brand A and 56 of brand B.

But is this easier?

The benefit here comes from having a single solution applicable to several variants of the same problem.

An agricultural wrinkle

In the above problem, suppose our *apple* grove wanted 720 pounds of nitrogen and 500 pounds of phosphoric acid, our *orange* grove wanted 640 pounds of nitrogen and 520 pounds of phosphoric acid, and our *lime* grove wanted 450 pounds of nitrogen and 300 pounds of phosphoric acid.

We *could* write, and solve, three separate systems of equations for these:

$$\begin{cases} 8x_a + 7y_a = 720 \\ 4x_a + 6y_a = 500 \end{cases} \qquad \begin{cases} 8x_o + 7y_o = 640 \\ 4x_o + 6y_o = 520 \end{cases} \qquad \begin{cases} 8x_\ell + 7y_\ell = 450 \\ 4x_\ell + 6y_\ell = 300 \end{cases}$$

But then we'd end up doing the same work three times!

Solving several related problems at once

Let's rewrite those three systems as:

$$\begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix} A = \begin{bmatrix} 720 \\ 500 \end{bmatrix} \qquad \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix} O = \begin{bmatrix} 640 \\ 520 \end{bmatrix} \qquad \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix} L = \begin{bmatrix} 450 \\ 300 \end{bmatrix}$$

And they have solutions:

$$A = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 720 \\ 500 \end{bmatrix} = \begin{bmatrix} 0.30 & -0.35 \\ -0.20 & 0.40 \end{bmatrix} \begin{bmatrix} 720 \\ 500 \end{bmatrix} = \begin{bmatrix} 41 \\ 56 \end{bmatrix}$$

$$O = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 640 \\ 520 \end{bmatrix} = \begin{bmatrix} 0.30 & -0.35 \\ -0.20 & 0.40 \end{bmatrix} \begin{bmatrix} 640 \\ 520 \end{bmatrix} = \begin{bmatrix} 10 \\ 80 \end{bmatrix}$$

$$L = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 450 \\ 300 \end{bmatrix} = \begin{bmatrix} 0.30 & -0.35 \\ -0.20 & 0.40 \end{bmatrix} \begin{bmatrix} 450 \\ 300 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

The only "hard part" is finding $\begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 0.30 & -0.35 \\ -0.20 & 0.40 \end{bmatrix}$, but once we do that once, we can use it over and over!

Warnings

Finding solutions with inverses *won't* work if the matrix in question has no inverse.

Uninvertable example

If we wanted to solve $\begin{bmatrix} 2 & 4 & 1 \\ 3 & -4 & 0 \end{bmatrix} X = \begin{bmatrix} 12 \\ -7 \end{bmatrix}$ we couldn't use $\begin{bmatrix} 2 & 4 & 1 \\ 3 & -4 & 0 \end{bmatrix}^{-1}$, which doesn't exist!

Under such a circumstance, we'd need to go back to good old-fashioned matrix reduction.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 12 \\ 3 & -5 & 0 & -7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0.2 & 1 \\ 0 & 1 & 0.15 & 2.5 \end{array}\right]$$

so $X = \begin{bmatrix} 1 - 0.2z \\ 2.5 - 0.15z \\ z \end{bmatrix}$ for any number z.