The Leontif demand model, and sketching inequalities

MATH 107: Finite Mathematics

University of Louisville

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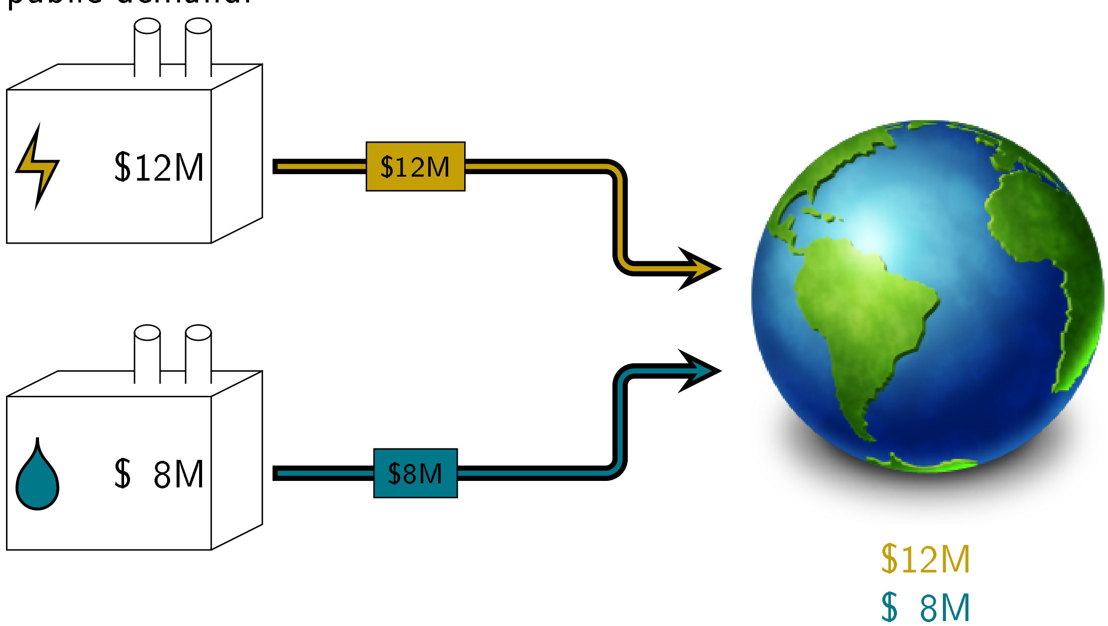
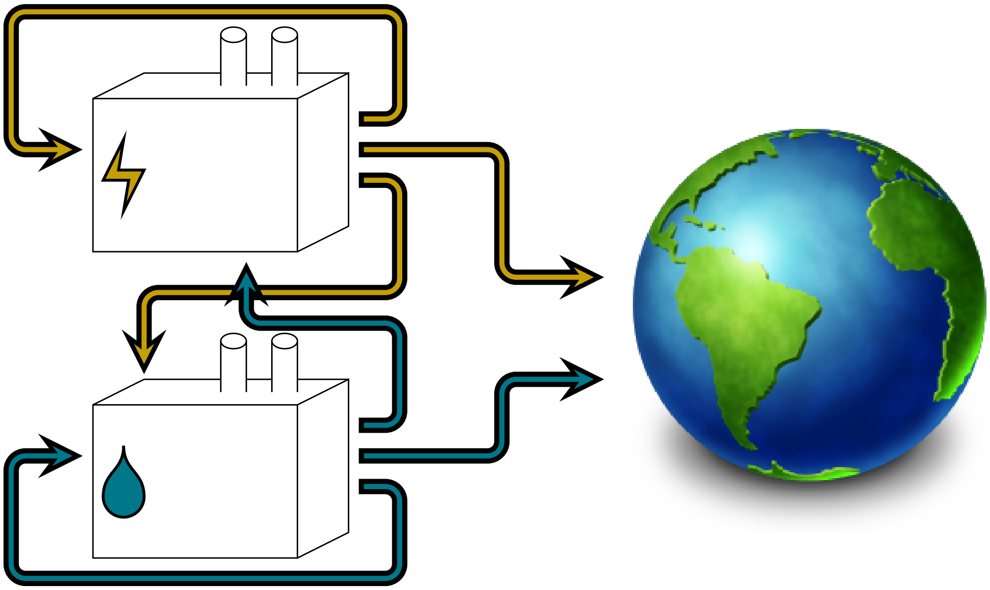
Basic economics of factories

A very simple economic model dictates that factories produce goods and services, and then provide them to the public, in response to public demand.



A more realistic picture

Factories do not *only* supply the public. They also serve each other. Furthermore, they consume their own product.



If we produce only enough to serve public demand, we will undershoot!

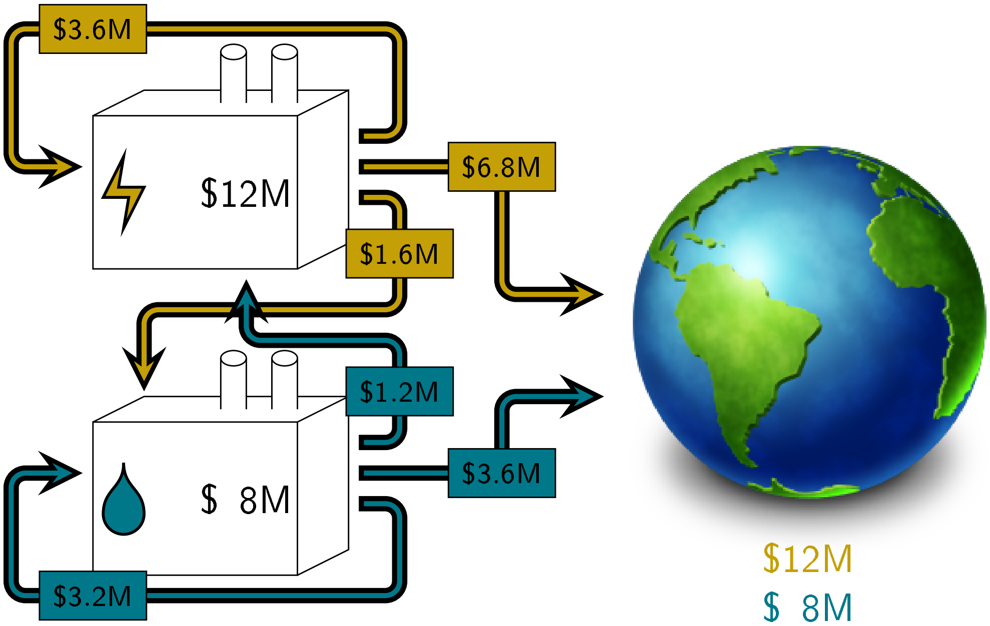
Specifying internal demand

A case study

As before, suppose the public wants \$12M worth of electricity and \$8M worth of water. But to produce \$1 of electricity requires \$0.30 of electricity and \$0.10 of water, while to produce \$1 worth of water requires \$0.20 of electricity and \$0.40 of water. How much should we produce of each?

We might explore by just *producing* \$12M in electricity and \$8M in water, but we know this won't work.

The wrong answer, investigated fully



\$12M in electricity needs \$3.6M in electricity, \$1.2M in water.
\$8M in water needs \$1.6M in electricity, \$3.2M in water.
Thus only \$6.8M in electricity and \$3.6M in water remain!

Names and abstractions

Let's try attaching names to things. We have three quantities worth considering:

External demand The public needs some amounts D_E and D_W of product.

Production We produce quantities P_E and P_W of product.

Internal demand Industries consume products in amounts proportional somehow to P_E and P_W.

Our problem statement told us that the internal demand of electricity is $0.3P_E + 0.2P_W$, while internal demand of water is $0.1P_E + 0.4P_W$. We can then describe internal demand with the *vector*

$$\begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \begin{bmatrix} P_E \\ P_W \end{bmatrix}$$

If we put all three of our quantities into vector form, we have an elegant relationship!

The matrix representation

We know that production should be the total of internal and external demand. If we have production and demand *vectors*, then:

$$\underbrace{\begin{bmatrix} P_E \\ P_W \end{bmatrix}}^{\text{Production}} = \overbrace{\underbrace{\begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix}}_{\text{Tech. matrix}} \begin{bmatrix} P_E \\ P_W \end{bmatrix}}^{\text{Internal demand}} + \overbrace{\begin{bmatrix} D_E \\ D_W \end{bmatrix}}^{\text{External demand}}$$

The *technology matrix* has as the entry in row i and column j the demands of industry j from industry i.
The entries of the technology matrix are all non-negative, and every column adds up to less than 1: why?

Thus, these industry problems boil down to solving equations of the form $P = MP + D$ for P.

How to solve this equation?

The \$20 million (in various products) question

If we have an equation of the form $P = MP + D$ with known M and D, how do we determine P?

We use a very slight variant of methods from the last section:

$$\begin{aligned} P = I_n P &= MP + D \\ I_n P - MP &= D \\ (I_n - M)P &= D \\ P &= (I_n - M)^{-1} D \end{aligned}$$

So in our original problem:

$$P = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 1.5 & 0.5 \\ 0.25 & 1.75 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} = \begin{bmatrix} 22 \\ 17 \end{bmatrix}$$

Moving to bigger industrial systems!

Nothing about the previous derivation requires just two industries. In fact, any square tech matrix and demand vector undergo the same process.

A three-industry model

Producing \$1 of agriculture uses \$0.20 of agriculture and \$0.40 of energy; producing \$1 of energy uses \$0.20 of energy and \$0.40 of widgets; producing \$1 of widgets uses \$0.10 of agriculture, \$0.10 of energy, and \$0.30 of widgets. How do we meet a public demand of \$20B in ag, \$10B in energy, and \$30B in widgets?

Here, $M = \begin{bmatrix} 0.2 & 0.0 & 0.1 \\ 0.4 & 0.2 & 0.1 \\ 0.0 & 0.4 & 0.3 \end{bmatrix}$ and $D = \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$.

Solving the Three-Industry System

Our scenario in matrices

$M = \begin{bmatrix} 0.2 & 0.0 & 0.1 \\ 0.4 & 0.2 & 0.1 \\ 0.0 & 0.4 & 0.3 \end{bmatrix}$ and $D = \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix}$.

As before, we find P to be $(I_3 - M)^{-1}D$. Note that

$$(I_3 - M)^{-1} = \begin{bmatrix} 0.8 & 0.0 & -0.1 \\ -0.4 & 0.8 & -0.1 \\ 0.0 & -0.4 & 0.7 \end{bmatrix}^{-1} = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix}$$

so:

$$P = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \\ 30 \end{bmatrix} = \begin{bmatrix} 33 \\ 37 \\ 64 \end{bmatrix}$$

Reusing solutions

Suppose that in the above scenario the energy needs of our society doubled. Would we have to start all over again from the beginning? No! In the equation $P = (I_3 - M)^{-1} D$, only D would change:

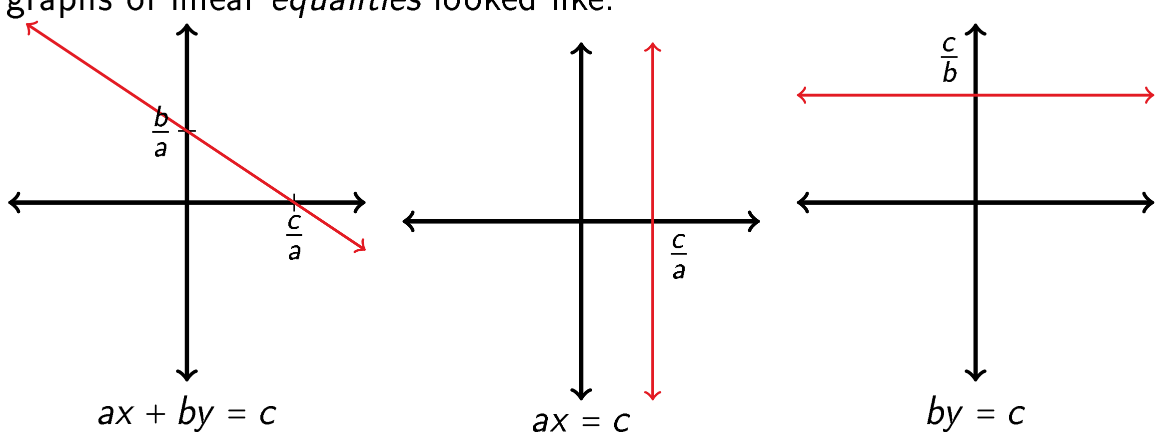
$$P = \begin{bmatrix} 1.3 & 0.1 & 0.2 \\ 0.7 & 1.4 & 0.3 \\ 0.4 & 0.8 & 1.6 \end{bmatrix} \begin{bmatrix} 20 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 34 \\ 51 \\ 72 \end{bmatrix}$$

so once you've set up a solution for a technology matrix M, it could work for several different demand matrices!

And now for something completely different. . .

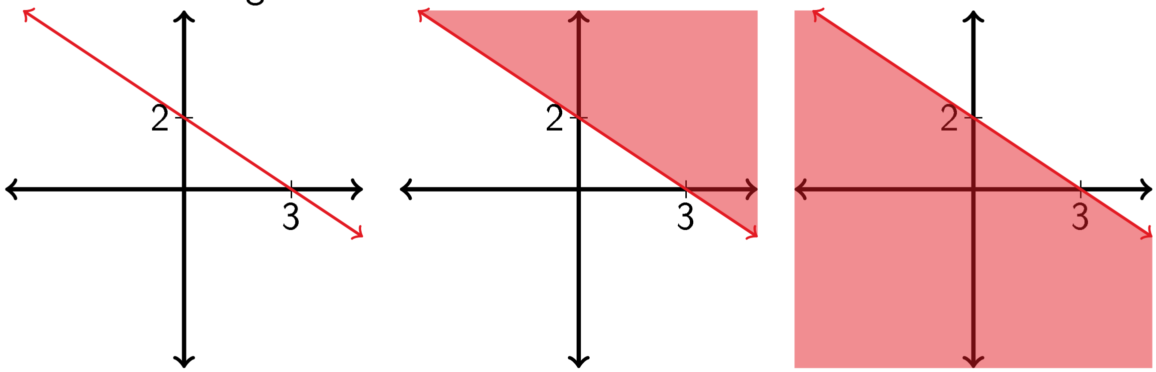
We now move to a new chapter, where we study linear *inequalities*.

The best way to understand these is graphically. Let's recall what our graphs of linear *equalities* looked like:



From equality to inequality in pictures

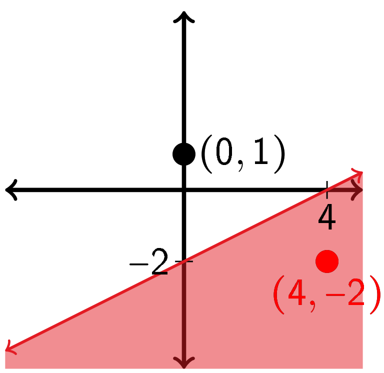
Once we have a linear *equality*, the graph of an associated inequality invovles shading one or the other side of the line.



$2x + 3y = 6$ $2x + 3y \geq 6$ $2x + 3y \leq 6$

Choosing sides

How do we know which side to use? We can probe with points.



$x - 2y \geq 4$

Does $(0, 1)$ satisfy our inequality? No!

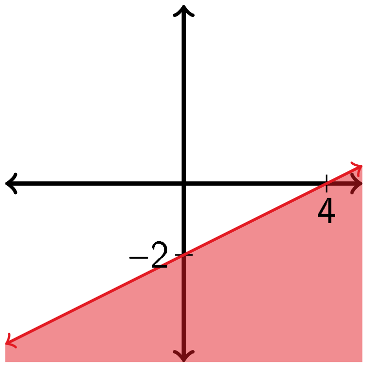
$$0 - 2(1) = -2 \not\geq 4$$

Does $(4, -2)$ satisfy our inequality? Yes!

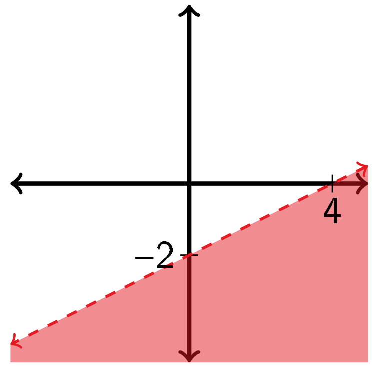
$$4 - 2(-2) = 8 \geq 4$$

Inclusive vs. exclusive inequalities

We use a dotted line to indicate an edge outside our solution set; this exhibits the difference between a *strict* and *nonstrict inequality*



$x - 2y \geq 4$



$x - 2y > 4$

Juggling multiple inequalities

Systems of inequalities can be used to describe many real world situations.

Sample scenario

I want to put no more than 40 pennies and nickels in a jar so that their total value is less than \$1.00. What combinations are possible?

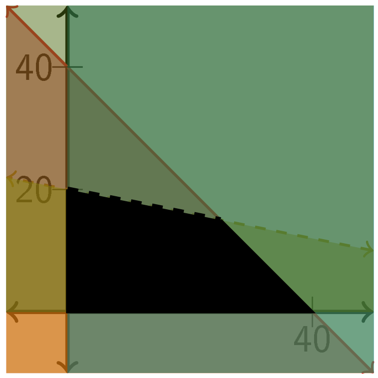
Let's denote by x the number of pennies and y the number of nickels. Then this situation is described by a system of four inequalities:

$$\begin{cases} x + y \leq 40 \\ x + 5y < 100 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

We can best visulaize this with a graph.

Graphing systems of inequalities

$$\begin{cases} x + y \leq 40 \\ x + 5y < 100 \\ x \geq 0 \\ y \geq 0 \end{cases}$$



The solution to a system of inequalities is the region where they *overlap*.