Linear programming

MATH 107: Finite Mathematics

University of Louisville

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A straightforward industrial problem

Here is a simple manufacturing question:

Cottage industry

I make designer hats and scarves part-time. I can make a hat from 240 yards of yarn and a scarf from 450 yards. A hat takes an hour and a half to make, and a scarf only an hour. Each week I have 21 hours and 7200 yards of yarn. I make a profit of $20 on each scarf and $16 on each hat. What combination of goods should I craft to maximize profit?

This problem is an example of what we will learn is called a *linear programming* problem.

What is "Linear Programming"?

In simple terms, an *optimization problem* consists of the following elements:

Variables	There are decisions to be made which can be numerically expressed (number of hats and number of scarves to produce per week).
Constraints	There are mechanical, legal, or other limitations on which combinations of these decisions are valid (amount of yarn and time available).
Objective	We have a numerical quantity, usually "profit" or "cost", which depends on our decisions and which we hope to maximize or minimize (profit from sale of a week's production of hats and scarves).

The simplest sort of optimization problem is one in which the *constraints* are linear inequalities and the *objective* is a linear function. This kind of optimization problem is a *linear program*.

Why is our cottage-industy problem a LP?

Parsing our problem

I make hats and scarves. A hat uses 240 yards of yarn and a scarf 450 yards. A hat takes 1.5 hours to make, and a scarf only an hour. Each week I have 21 hours and 7200 yards of yarn. I make a profit of \$20 on each scarf and \$16 on each hat.

Let's find the variables, constraints, and an objective above.

- Variables: Let x be the # of hats made, and y the # of scarves.
- Constraint: I use $240x + 450y$ yards of yarn to make these products. I have 7200 yards of yarn in total to use.
- Constraint: It takes $1.5x + y$ hours to make these products. I have 21 hours in total to use.
- Goal: The profit to be maximized is $16x + 20y$.

$$\begin{cases} 240x + 450y \leq 7200 \\ 1.5x + y \leq 21 \end{cases}$$

Solving a LP problem, step 1: abstraction

Still the same cottage industry

I make hats and scarves. A hat uses 240 yards of yarn and a scarf 450 yards. A hat takes 1.5 hours to make, and a scarf only an hour. Each week I have 21 hours and 7200 yards of yarn. I make a profit of $20 per scarf and $16 per hat. How do I maximize profit?

On the last slide we took the first step: abstracting the problem.

The same thing, without words

x is the number of hats, y the number of scarves.

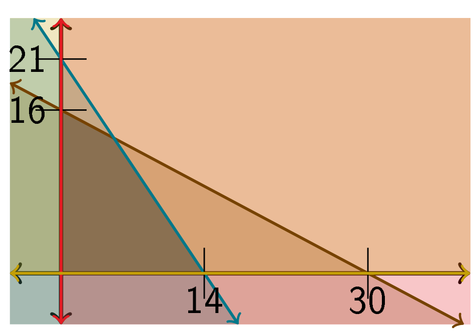
$$\text{Constraints:}\begin{cases} 240x + 450y \leq 7200 \\ 1.5x + y \leq 21 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Objective: maximize $16x + 20y$.

Solving a LP problem, step 2: sketching constraints

$$\begin{cases} 240x + 450y \leq 7200 \\ 1.5x + y \leq 21 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

We know, from the last lesson, that these constraints describe a section of the plane.



The area satisfying the constraints is called the *feasible region*.

A critical property of linear programs

Once we have a feasible region, optimizing our objective function is not too hard, due to a very useful fact about LPs:

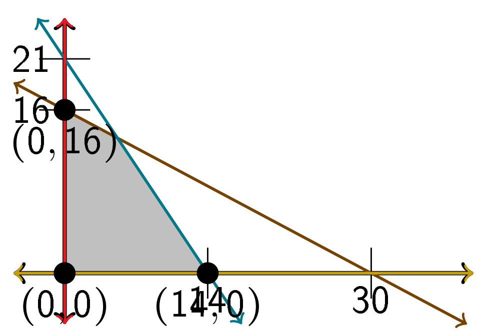
The Fundamental Theorem of Linear Programming

The objective function is maximized (or minimized) at a *corner of the feasible region*.

So to find an optimal point, we simply need to consider each of the corners of the region!

Solving a LP problem, step 3: finding corners (cont'd)

$$\begin{cases} 240x + 450y \leq 7200 \\ 1.5x + y \leq 21 \\ x, y \geq 0 \end{cases}$$



Three of the corners have obvious coordinates.
The fourth, though, is difficult.

Solving a LP problem, step 3: finding corners

Our mysterious fourth corner is at the intersection of the two lines:

$$\begin{cases} 240\,x + 450y = 7200 \\ 1.5x + y = 21 \end{cases}$$

We can find this with any of our system-solving techniques!

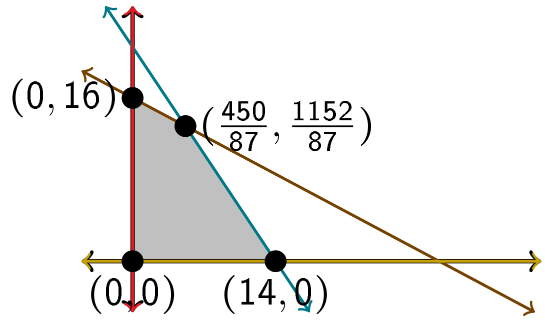
$$y = 21 - 1.5x$$

$$\begin{aligned} 240x + 450(21 - 1.5x) &= 7200 \\ 240x + 9450 - 675x &= 7200 \\ -435x &= -2250 \\ x &= \frac{2250}{435} = \frac{450}{87} \end{aligned}$$

so $y = 21 - 1.5 \cdot \frac{450}{87} = \frac{1152}{87}$.

Solving a LP problem, step 4: testing corners

Now we know our objective function $16x + 20y$ is maximized at some corner. Let's try each.



Profit at $(0, 0)$: $\$16 \cdot 0 + \$20 \cdot 0 = \$0$
Profit at $(14, 0)$: $\$16 \cdot 14 + \$20 \cdot 0 = \$224$.
Profit at $(0, 16)$: $\$16 \cdot 0 + \$20 \cdot 16 = \$320$.
Profit at $(\frac{450}{87}, \frac{1152}{87})$: $\$16 \cdot \frac{450}{87} + \$20 \cdot \frac{1152}{87} \approx \$347.59$.
Thus, our best strategy is to make about $\frac{450}{87}$ hats and $\frac{1152}{87}$ scarves per week.

Redoing a problem with a new objective

Suppose that my profit on scarves drops to \$10. Would I have to start my analysis over? No! The variables and constraints are still the same, so my feasible region has the same four corners.
So we just test the same four potential optima with the new objective function $16x + 10y$:

Profit at $(0, 0)$: $\$16 \cdot 0 + \$10 \cdot 0 = \$0$
Profit at $(14, 0)$: $\$16 \cdot 14 + \$10 \cdot 0 = \$224$.
Profit at $(0, 16)$: $\$16 \cdot 0 + \$10 \cdot 16 = \$160$.
Profit at $(\frac{450}{87}, \frac{1152}{87})$: $\$16 \cdot \frac{450}{87} + \$10 \cdot \frac{1152}{87} \approx \$215.17$.

In this scenario, it makes more sense to produce as many hats as possible (i.e. 14 hats and no scarves).

Another LP example

The same techniques suffice whenever you have *constraints* and a *goal*.

A little bit louder and a little bit worse

You are managing an investment to be split among mutual funds which pay 1.5% and a CD which pays 1%; you have a total of \$100,000 to invest. To keep the investment low-risk, you are required to put at least \$10,000 into the CD, and no more than twice as much into the mutual fund as the CD. How do you maximize interest?

We have two variables: let x and y represent our investment in the CD and the mutual fund respectively. Now we have constraints:

$$\begin{cases} x + y \leq 100000 \\ x \geq 10000 \\ y \geq 0 \\ y \leq 2x \end{cases}$$

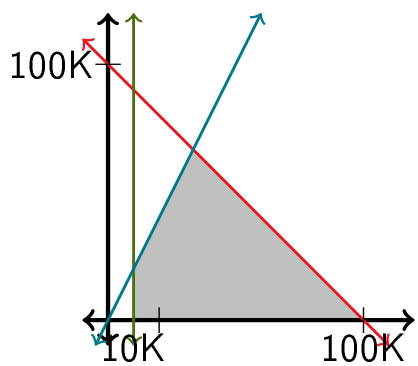
We have an objective: to maximize the interest $0.01x + 0.015y$.

Graphing the LP

Our LP problem

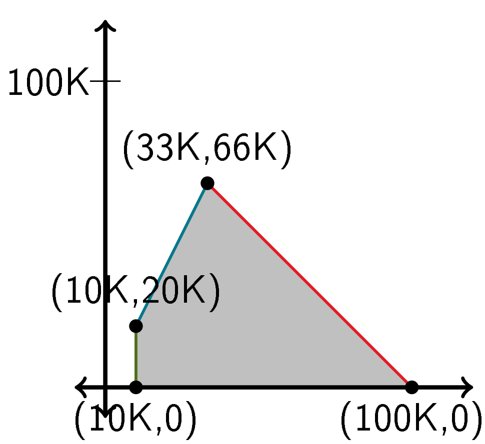
Maximize $0.01x + 0.015y$ subject to $\begin{cases} x + y \leq 100000 \\ x \geq 10000 \\ y \geq 0 \\ y \leq 2x \end{cases}$

We draw the feasible region:



Identifying corners

$$\begin{cases} x + y \leq 100000 \\ x \geq 10000 \\ y \geq 0 \\ y \leq 2x \end{cases}$$



Finding the best corner

Our objective function (for interest earned on our investment) is $0.01x + 0.015y$.

Interest at $(10000, 0)$: $0.01 \cdot 10000 + 0.015 \cdot 0 = 100$
Interest at $(10000, 20000)$: $0.01 \cdot 10000 + 0.015 \cdot 20000 = 400$
Interest at $(33333, 66666)$: $0.01 \cdot \frac{100000}{3} + 0.015 \cdot \frac{200000}{3} \approx 1333$
Interest at $(1000000, 0)$: $0.01 \cdot 100000 + 0.015 \cdot 0 = 1000$

So our best choice is to invest the whole bundle with a third in the CD and two-thirds in mutuals, to earn \$1333 in interest.