More linear programming, and Logic

MATH 107: Finite Mathematics

University of Louisville

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An abstract linear program

Sometimes there's no story, just constraints and an objective function:

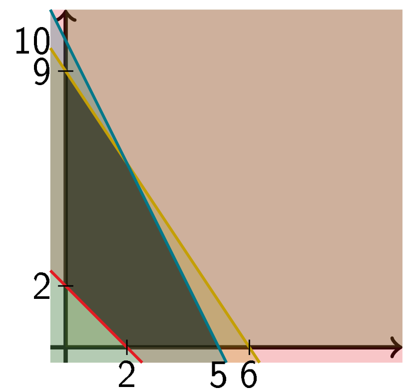
The whole question

Find the minimum and maximum values of $2x - y$ on $\begin{cases} x + y \geq 2 \\ 6x + 4y \leq 36 \\ 4x + 2y \leq 20 \\ x, y \geq 0 \end{cases}$

A question like this is just like an LP in words, except the abstraction's already been done!

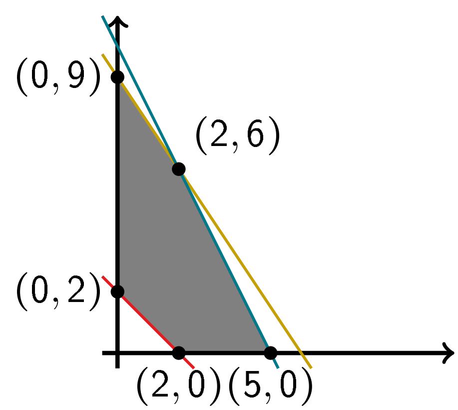
Drawing the feasible region

$$\begin{cases} x + y \geq 2 \\ 6x + 4y \leq 36 \\ 4x + 2y \leq 20 \\ x, y \geq 0 \end{cases}$$



Finding the corners

$$\begin{cases} x + y \geq 2 \\ 6x + 4y \leq 36 \\ 4x + 2y \leq 20 \\ x, y \geq 0x, y \geq 0 \end{cases}$$



Four of our corners are on the axes and will be easy.

The last corner is the intersection of $6x + 4y = 36$ and $4x + 2y = 20$

That point would be $(2, 6)$.

Testing the corners

We know the minimum and maximum of the objective function $2x - y$ occur on the corners, so let's test it:

At $(0, 2)$: $2x - y = 2 \cdot 0 - 2 = -2$.
At $(0, 9)$: $2x - y = 2 \cdot 0 - 9 = -9$.
At $(2, 6)$: $2x - y = 2 \cdot 2 - 6 = -2$.
At $(5, 0)$: $2x - y = 2 \cdot 5 - 0 = 10$.
At $(2, 0)$: $2x - y = 2 \cdot 2 - 0 = 4$.

So our objective function is minimized at $(0, 9)$ and maximized at $(5, 0)$.

An example unbounded above

An explicit problem: agriculture again!

Each bag of mix A of plant food costs \$30 and contains 20 pounds of phosphoric acid, 30 pounds of nitrogen, and 5 pounds of potash. Each bag of mix B costs \$35 and contains 10 pounds of phosphoric acid, 30 pounds of nitrogen, and 10 pounds of potash. If we need at least 460 pounds of phosphoric acid, 960 pounds of nitrogen, and 220 pounds of potash, what is the cheapest combination we can buy?

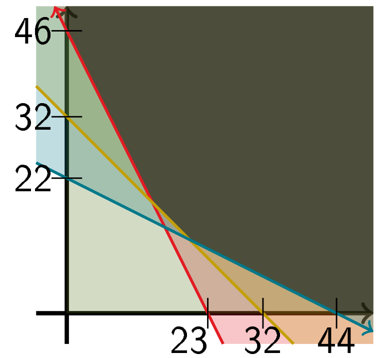
Let x and y represent the number of bags of mix A and B respectively. Now we have constraints born of our nutritional requirements:

$$\begin{cases} 20x + 10y \geq 460 \\ 30x + 30y \geq 960 \\ 5x + 10y \geq 220 \\ x, y \geq 0 \end{cases}$$

and an objective to minimize $30x + 35y$.

A different type of feasible region

$$\begin{cases} 20x + 10y \geq 460 \\ 30x + 30y \geq 960 \\ 5x + 10y \geq 220 \\ x, y \geq 0 \end{cases}$$



Notably, this feasible region is *unbounded*; we might need to consider the possibility that x, or y, or both, are very large.

Finding the corners

$$\begin{cases} 20x + 10y \geq 460 \\ 30x + 30y \geq 960 \\ 5x + 10y \geq 220 \\ x, y \geq 0 \end{cases}$$

Two corners are easy; the others are intersection points.

$20x + 10y = 460$ and $30x + 30y = 960$ intersect at $(14, 18)$.
$30x + 30y = 960$ and $5x + 10y = 220$ intersect at $(20, 12)$.

Testing the corners of an unbounded region

At $(0, 46)$:	$30x + 35y = 1510$.
At $(14, 18)$:	$30x + 35y = 1050$.
At $(20, 12)$:	$30x + 35y = 1020$.
At $(0, 44)$:	$30x + 35y = 1540$.
At (big, 0):	$30x + 35y$ is big.
At (0, big):	$30x + 35y$ is big.
At (big, big):	$30x + 35y$ is big.

The unboundedness presents a bit of a problem: what does it mean to "test" the top, right, and upper-right sections?

We'll think of them as just "very large", and test all 7 "corners" with our objective function $30x + 35y$.

Since we seek to minimize, our large values aren't worth considering at all, and our minimizing choice will be 20 bags of mix A, 12 of mix B.

Another different problem

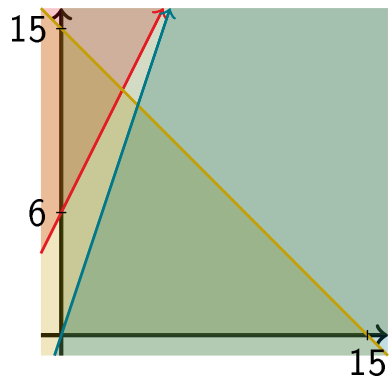
Why should this be interesting?

Find the minimum and maximum values of $4x + 3y$ on $\begin{cases} -2x + y \geq 6 \\ x + y \leq 15 \\ 3x - y \geq 0 \\ x, y \geq 0 \end{cases}$

Let's look at its feasible region!

An unusual feasible region

$$\begin{cases} -2x + y \geq 6 \\ x + y \leq 15 \\ 3x - y \geq 0 \\ x, y \geq 0 \end{cases}$$



There is no feasible region, so this LP has no solution!

Possible sizes of LP systems

The techniques we've seen to solve 2-variable LPs work no matter how many constraits we have.

Is there a way to consider linear programs with *three or more variables*?

A linear inequality in three variables is a division of 3-dimensional space, and we could do the same thing with these inequalities in space as we do with 2-variable inequalities in the plane.

Because it's difficult to visualize, though, we won't. Look at Chapter 6 for gory details.

What is logic?

Logic is the branch of mathematics that deals with assessing the truth of complicated statements based on the truth of simpler statements.

Simple statements

- $2 + 2 = 5$. (false)
- Frankfort is the capital of Kentucky. (true)
- My brother has blue eyes. (true, but you don't know)
- $x^2 > 16$. (truth depends on x)

Note that some statements are true, and some are false, and some don't have easily determined truth values.

How can we use logic?

Logic lets us assemble larger statements out of smaller ones by connecting them together.

Compound statements

- Louisville is the capital of Kentucky **or** Nashville is the capital of Tennessee. (true)
- Mary is **not** John's daughter. (unknown)
- x is a positive number **and** x is a negative number. (false)
- **If** x is a positive number, **then** $x + 1$ is a positive number. (true)

The compounds "and", "or", "not", "if" function on *statements* like operations like addition or multiplication function on *numbers*.

Compounding statements of known truth is easy.

But sometimes the truth of a compound doesn't depend on the truth value of its parts—just like $x \cdot 0 = 0$ or $\frac{x}{x} = 1$ no matter what x is.

Logic as an algebra

We can represent simple statements as named abstractions: "p", "q", or "r". Each letter represents a statement (either true or false). We build compound statements with operations of *propositional calculus*.

Propositional operations

- $\neg p$ is the compound statement "p is not true".
- $p \vee q$ is the compound statement "p or q".
- $p \wedge q$ is the compound statement "p and q".
- $p \rightarrow q$ is the compound statement "if p, then q".

We can build more complicated combinations of these various operations, like

$$[(p \vee q) \wedge \neg r] \rightarrow (q \wedge r)$$

which translates to "if p or q is true and r is false, then both q and r are true".