Logic, continued

MATH 107: Finite Mathematics

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Truth tables

To describe how a logical operation (or compound statement) works, we use a structure called a *truth table*, which lists all possible combinations of truth values of named statements.
If we have 1 named statement, there are 2 possibilities.
If we have 2 named statements, there are 4 possibilities.
If we have 3 named statements, there are 8 possibilities.

p
T
F

p	q
T	T
T	F
F	T
F	F

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

How to reliably build a truth table

- Write out your n named statements at the top of each column, and leave 2^n spaces for truth values.
- Alternate between the two truth values in one column.
- Alternate, repeating each truth value twice, in another column.
- Alternate, repeating each value four times, in another column.
- Continue doubling repetition counts, until all columns are filled.

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Why use truth tables?

Recall that named statements could represent either false or true specific statements—we don't know which!

A truth table gives us the ability to investigate what happens in all possible scenarios.

Truth tables will also give us a complete understanding of how individual logical operations work.

How *not* to study logic

The simplest operation is the *negation*, performed by the operator $\neg$, pronounced “not”. $\neg p$ could be read “p is false” or “p is not true”. We can investigate how it works with a truth table:

Example p: “$2 + 2 = 4$”.
$\neg p$: “$2 + 2 \neq 4$”.
Example p: “Louisville is in California”.
$\neg p$: “Louisville is not in California”.

p	$\neg p$
T	F
F	T

So negation just “flips” truth and falsity of statements.

Conjunction junction, what's your function

Another easily understood operation is *conjunction*, performed by the operator $\wedge$, pronounced “and”. $p \wedge q$ could be read “p and q”. This too is explorable with a truth table:

p: “$2 + 2 = 4$”. q: “Louisville is in Kentucky”.
$p \wedge q$: “$2 + 2 = 4$ and Louisville is in Kentucky”.
p: “$2 + 2 = 4$”. q: “Louisville is in Indiana”.
$p \wedge q$: “$2 + 2 = 4$ and Louisville is in Indiana”.
p: “$2 + 2 = 5$”. q: “Louisville is in Kentucky”.
$p \wedge q$: “$2 + 2 = 5$ and Louisville is in Kentucky”.
p: “$2 + 2 = 5$”. q: “Louisville is in Indiana”.
$p \wedge q$: “$2 + 2 = 5$ and Louisville is in Indiana”.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

A conjunction is true when *both* its sub-propositions are true.

Disjunction junction, what's your. . . dysfunction?

The third common operator is *disjunction*, performed by the operator $\vee$, pronounced "or", so we read $p \vee q$ as "p or q". This will differ in one respect from common usage:

p: "Cats are animals". q: "$5 > 2$".
$p \vee q$: "Cats are animals or $5 > 2$".
p: "Cats are animals". q: "$0 > 1$".
$p \vee q$: "Cats are animals or $0 > 1$".
p: "Cats are vegetables". q: "$5 > 2$".
$p \vee q$: "Cats are vegetables or $5 > 2$".
p: "Cats are vegetables". q: "$0 > 1$".
$p \vee q$: "Cats are vegetables or $0 > 1$".

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

A disjunction is true when *at least one* its sub-propositions is true.

Chains of implication

The *implication* or *conditional* gives students the most trouble. Its associated operator is $\rightarrow$, pronounced "implies" or "only if" or "if . . . then". We may find ourselves applying this in unfamilar ways.

p: "Louisville is in KY". q: "UofL is in KY".
$p \rightarrow q$: "If Louisville is in Kentucky, so is UofL.".
p: "Louisville is in KY". q: "UofL is in HI".
$p \rightarrow q$: "If Louisville is in KY, then UofL is in HI".
p: "-1 is positive". q: "$-1 + 2$ is positive".
$p \rightarrow q$: "If -1 is positive, then so is $-1 + 2$".
p: "Louisville is in Hawaii". q: "UofL is in Hawaii".
$p \rightarrow q$: "If Louisville is in Hawaii, then so is UofL".

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The implication continued

Is this statement true?

"If Louisville is in Florida, then UofL is in Wisconsin."

Let's look at a truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Be warned: mathematical implication doesn't mean two statements have to be *related* in any way!

A paraphrase from the syllabus

"If you get 90% of the points, you will get a letter grade of at least A– or better."

There are four possibilities alluded to here, and only one of them is off the table.

A variation on implication

One important distinction between implication and the other operations is that *order matters*.

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

"If Ginny is a cat, then Ginny is an animal" is incontestably true; "If Ginny is an animal, then Ginny is a cat" could quite reasonably not be true.

$q \to p$ is known as the *converse* of $p \to q$.
Two other variations are the *inverse* ($\neg p \to \neg q$) and the *contrapositive* ($\neg q \to \neg p$).

Complicated compositions

A logical statement may consist of many operations performed end-to-end. How do we find the truth value of, for instance: $(p \vee q) \rightarrow [(\neg p) \wedge q]$?
We break it down into its constituent parts for truth-table analysis. $p \vee q$ and $\neg p$ are easy. From $\neg p$ we can find $(\neg p) \wedge q$. Finally, we can take an implication involving the two largest things we've found.

p	q	$p \vee q$	$\neg p$	$\neg p \wedge q$	$(p \vee q) \rightarrow [(\neg p) \wedge q]$
T	T	T	F	F	F
T	F	T	F	F	F
F	T	T	T	T	T
F	F	F	T	F	T

By methodically working from the inside out, we can consider expressions of any size!

A note on precedence

We're used to precedence rules in arithmetic: in $3 \cdot 4 + 5$ we do the multiplication first and then the addition; if we wanted to add 4 to 5 and then multiply by 3, we'd write $3 \cdot (4 + 5)$.
Similar rules apply for logical operations. The precedence order is:

- negation ($\neg$)
- conjunction and disjunction ($\wedge$ and $\vee$; use parentheses if using both to clarify order)
- implication ($\rightarrow$)

So, for instance, $\neg p \wedge \neg q$ would be understood as $(\neg p) \wedge (\neg q)$, while $p \vee q \rightarrow p$ would be $(p \vee q) \rightarrow p$.

When in doubt, err on the side of using parentheses to make your point clear!

Contradictions

Some compounds are false regardless of their underlying statements. Here's a simple one: $(\neg p \vee \neg q) \wedge (p \wedge q)$.

p	q	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$p \wedge q$	$(\neg p \vee \neg q) \wedge (p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	T	F	F

Tautologies

Likewise, some compounds are true regardless of their underlying statements. Here's one: $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	Orig.
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Such a statement is called a "tautology". The above tautology guarantees that "if p implies q and q implies r, then p implies r".

Return to implications

Recall that there are three variants on the implication $p \to q$: the *converse* $q \to p$, the *inverse* $\neg p \to \neg q$ (which is rarely used), and the *contrapositive* $\neg q \to \neg p$.

We saw that an implication and its converse are quite different statements!

However, an implication and its contrapositive always have the same truth value; we call such statements *logically equivalent*.

p	q	$p \to q$	$q \to p$	$\neg q \to \neg p$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Contrapositive examples

Let's look at a few incontestably true statements we've seen in the course and consider their contrapositives:

If Ginny is a cat, then Ginny is an animal.

Contrapositive: "If Ginny is not an animal, then Ginny is not a cat."

If you get at least 90%, your letter grade will be at least an A–.

Contrapositive: "Your letter grade will be less than an A— only if you get less than 90%."

If x is positive, then so is $x + 1$.

Contrapositive: "If $x + 1$ is not positive, then x isn't positive either."

A few more logical equivalencies

You don't need to know these, but they may be useful to you.

Double negation	$\neg(\neg p) \equiv p$
Absorption of $\vee$	$p \vee p \equiv p$
Absorption of $\wedge$	$p \wedge p \equiv p$
Commutativity of $\vee$	$p \vee q \equiv q \vee p$
Commutativity of $\wedge$	$p \wedge q \equiv q \wedge p$
Associativity of $\vee$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Associativity of $\wedge$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Redefinition of $\rightarrow$	$p \rightarrow q \equiv \neg p \vee q$
DeMorgan's laws	$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$ $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$
Distributivity	$(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$ $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$