Sets

MATH 107: Finite Mathematics

University of Louisville

February 24, 2014

What is a set?

A set is an *unordered* structure containing any number of *distinct* objects.

These objects (called *elements* of the set) can be anything: numbers, words, people, functions — even other sets!

Conventionally we label a set with a *capital letter*, and list its elements in curly braces.

Several descriptions of the same set

- $A = \{1, 3, 4\}$.
- A is the set with elements 1, 3, and 4.
- $A = \{4, 1, 3\}$.
- $A = \{3, 1, 3, 4, 1, 1\}$.

Describing sets by patterns

Often it is tedious to list out all the elements of a set, so we might use ellipses to describe that many omitted elements follow a certain pattern.

For instance, we might write $\{1, 2, 3, \ldots, 50\}$ instead of actually writing out a set containing the first 50 positive integers.

Any pattern of membership can be shortened using ellipsis once the pattern is clear.

Many sets described by patterns

- $A = \{-21, -16, -11, \ldots, 44\}$.
- $B = \{2, 3, 5, 7, 11, 13, 17, \ldots, 97\}$.
- $C = \{1, 2, 4, 8, 16, \ldots, 1024\}$.
- $D = \{\text{Washington}, \text{Adams}, \text{Jefferson}, \ldots, \text{Obama}\}$.

Infinite sets

The pattern-following description becomes particularly useful when our list would need *infinitely* many entries. Compare the two following descriptions:

$$A = \{1, 3, 5, 7, \ldots, 99\} \text{ and } B = \{1, 3, 5, 7, \ldots\}$$

Note that A constists only of the 50 odd positive numbers less than 100, while B contains *every* odd positive number. We can describe infinite sets with patterns too:

Examples of infinite sets

- $S = \{1, 4, 9, 16, 25, \ldots\}$.
- $T = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$.
- $X = \{3, 7, 11, 15, 19, \ldots\}$.
- $Y = \{\ldots, -9, -5, -1, 3, 7, 11, 15, 19, \ldots\}$.

Describing sets by rules

Instead of using an ellipsis-described pattern, we can use a rule instead, with a specific format:

$$S = \{\langle variable\rangle | \langle condition\rangle\}$$

This template isn't obvious, so let's look at some examples:

How rules might describe a set

- $\{x|x \text{ is a positive integer}\} = \{1, 2, 3, \ldots\}$.
- $\{t|t^3 - t = 0\} = \{-1, 0, 1\}$.
- $\{k|k \text{ is a president of the U.S.}\} = \{\text{Washington}, \text{Adams}, \text{Jefferson}, \ldots, \text{Obama}\}$.

Membership in sets

For any object x and set A, either x is an element of A or it is not. We denote these two possibilities with the symbolic forms $x \in A$ and $x \notin A$ respectively.

Examples of membership

- $2 \notin \{1, 3, 4\}$.
- For $A = \{1, 5, 6, 10\}$, $5 \in A$.
- $18 \in \{2, 4, 6, 8, \ldots, 100\}$.
- For $B = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$, $\frac{5}{7} \notin B$.

A special set or two

One set is a rather unusual one: a set can contain any number of elements, including none at all.

Definition

The *empty set* or *null set*, denoted $\varnothing$ or $\{\}$, is the set containing no elements.

Another special set is somewhat context-dependent: the *universal set*, denoted U, is the set of all "objects of interest". What objects are "interesting" depends on the sets you're looking at:

Universal set examples

- If $A = \{-21, -16, -11, \ldots, 44\}$, then U might contain all integers.
- If $B = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$, then U might contain all rational numbers.
- If $C = \{\text{Washington}, \text{Adams}, \text{Jefferson}, \ldots, \text{Obama}\}$, then U might contain all people, living or dead.

Introduction of set operations

As with logical statements, sets can be put together using some novel operations. These operations will be as follows:

- The *union* of two sets.
- The *intersection* of two sets.
- The *complement* of a set.

We will discuss each of these operations individually.

Set unions

Definition

The *union* of sets A and B, denoted $A \cup B$, is the set of all objects which are in either A or B or both.

So, for instance, if $A = \{1, 3, 5\}$, $B = \{1, 2, 4\}$, and $C = \{2, 4, 6, 8, \ldots\}$:

- $A \cup B = \{1, 2, 3, 4, 5\}$.
- $A \cup C = \{1, 2, 3, 4, 5, 6, 8, 10, 12, \ldots\}$.
- $B \cup C = \{1, 2, 4, 6, 8, 10, 12, \ldots\}$.
- $A \cup \varnothing = \{1, 3, 5\}$.

Note the symbolic similarity of $A \cup B$ (the set of things in A *or* B) to $p \vee q$ (the statement which is true when p *or* q is true). This is made explicit by a rule-based definition of the union:

Definition (Alternative definition of the union)

$$A \cup B = \{x | (x \in A) \vee (x \in B)\}$$

Set intersections

Definition

The *intersection* of sets A and B, denoted $A \cap B$, is the set of all objects which are in both A and B.

So, for instance, if $A = \{1, 3, 5\}$, $B = \{1, 2, 4\}$, and $C = \{2, 4, 6, 8, \ldots\}$:

- $A \cap B = \{1\}$.
- $A \cap C = \varnothing$.
- $B \cap C = \{2, 4\}$.
- $C \cap \varnothing = \varnothing$.

Note the symbolic similarity of $A \cap B$ (the set of things in both A *and* B) to $p \wedge q$ (the statement which is true when p *and* q are true). This is made explicit by a rule-based definition of the intersection:

Definition (Alternative definition of the intersection)

$$A \cap B = \{x | (x \in A) \wedge (x \in B)\}$$

Set complements

Definition

The *complement* of a set A, denoted A' is the set of all objects which are *not* in A.

Note that we need a "universal set" U to describe "all objects".
For instance, if $A = \{1,3,5\}$, $B = \{1,2,4\}$, $C = \{2,4,6,\ldots\}$, and $U = \{1,2,3,\ldots\}$:

- $A' = \{2,4,6,7,8,9,\ldots\}$.
- $B' = \{3,5,6,7,8,9,\ldots\}$.
- $C' = \{1,3,5,7,9,11,\ldots\}$.

Note the symbolic similarity of A' (the set of things *not* in A) to $\neg p$ (the statement which is true when p is *not* true). This is made explicit by a rule-based definition of the complement:

Definition (Alternative definition of the complement)

$$A' = \{x | \neg(x \in A)\}$$

Tests on sets

There are a few important questions to be asked about sets:

- Is a given object in a set A?
- Does a particular set A lie entirely within another set B?
- Are two sets identical?

The first we have already seen; the second and third we will describe in greater detail.

Subsets

Definition

A is a *subset* of B, denoted $A \subset B$, if every element of A is an element of B.

For instance, if $A = \{1,3,5\}$, $B = \{1,2,4\}$, $C = \{2,4,6,8,\ldots\}$, and $U = \{1,2,3,\ldots\}$, we might note some subset inclusions:

- $\varnothing \subset A$, $\varnothing \subset B$, and $\varnothing \subset C$.
- $A \subset U$, $B \subset U$, and $C \subset U$.
- $A \not\subset C$, $B \not\subset C$, $A \not\subset B$.
- $A \subset C'$.

This concept is *slightly* connected to the idea of implication: the statement "$A \subset B$" is true only when "$x \in A \implies x \in B$" is always true.

Equality

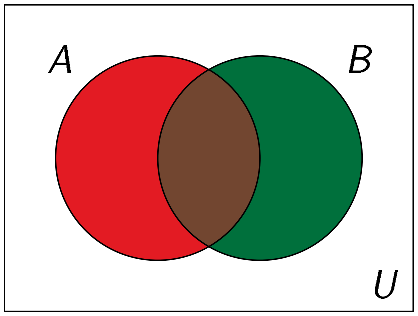
Definition

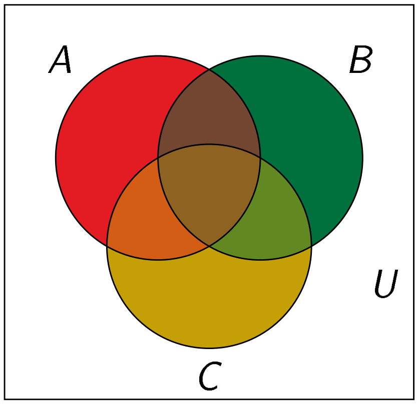
A is equal to B, denoted $A = B$, if A and B have the exact same elements.

In particular, the statement that $A = B$ is the same as claiming that both $A \subset B$ and $B \subset A$.

Venn diagrams

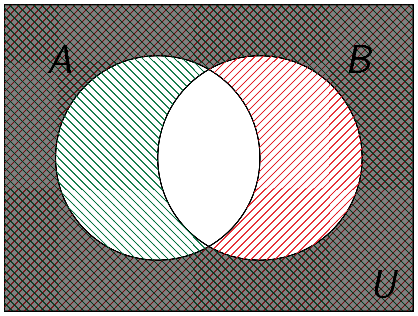
To visualize a set and operations on it, the most common tool is a *Venn diagram*, which displays regions associated with every possible set membership.



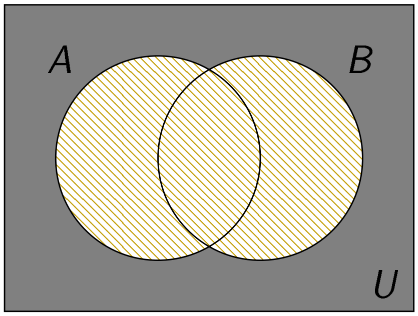


Set operations visualized with Venn diagrams

Using Venn diagrams, we can specifically consider regions associated with specific set operations.



$A' \cap B'$



$(A \cup B)'$

We may thus conclude, from this picture, that $A' \cap B' = (A \cup B)'$.

Other rules governing set operations

All of these could be proven by drawing pictures, but we aren't going to.

Double complement	$(A')' = A$
Absorption of $\cup$	If $B \subset A$, then $A \cup B = A$.
Absorption of $\cap$	If $B \subset A$, then $A \cap B = B$.
Commutativity of $\cup$	$A \cup B = B \cup A$.
Commutativity of $\cap$	$A \cap B = B \cap A$.
Associativity of $\cup$	$(A \cup B) \cup C = A \cup (B \cup C)$
Associativity of $\cap$	$(A \cap B) \cap C = A \cap (B \cap C)$
DeMorgan's laws	$(A' \cup B') = (A \cap B)'$ $(A' \cap B') = (A \cup B)'$
Distributivity	$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

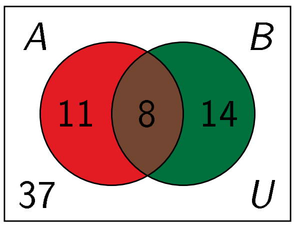
Using Venn diagrams for counting purposes

We can use a Venn diagram to understand the *sizes* of sets.

A sample question

There are 70 students in this class. 19 of them have a grade of A, 22 are first-years, and 8 first-years have an A. How many students are neither first-year nor currently getting an A?

Let U be all the students, A those getting an A, and B the first-years.



We know $A \cap B$ has 8 elements. A has 19 elements, 8 of which are in $A \cap B$. B has 22 elements, 8 of which are in $A \cap B$. U has 70 elements, 33 of which are in $A \cup B$.