Principles of Counting

MATH 107: Finite Mathematics

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Notation for counting elements of sets

We let $n(A)$ denote the number of elements in a finite set A.

Examples

If $A = \{1, 2, 3, 4\}$, $B = \{0\}$, and $C = \{0, 2, 4, 6, 8\}$, then:

- $n(A) = n(\{1, 2, 3, 4\}) = 4$
- $n(B) = n(\{0\}) = 1$
- $n(C) = n(\{0, 2, 4, 6, 8\}) = 5$
- $n(A \cap B) = n(\varnothing) = 0$
- $n(A \cup B) = n(\{0, 1, 2, 3, 4\}) = 5$
- $n(A \cup C) = n(\{0, 1, 2, 3, 4, 6, 8\}) = 7$

Addition principle

If we have sets A and B of two similar types of objects, we might wonder how many objects there are among both sets, i.e., in $A \cup B$.

If A and B have no elements in common, we call them *disjoint* and then we can just add up their individual counts.

Specialized addition principle

If $A \cap B = \varnothing$, then
$$n(A \cup B) = n(A) + n(B)$$

For example, suppose that this class contains 22 first-years and 16 second-years. How many students in the class are in their first two years of university? $22 + 16 = 38$ in total, because these two sets of students are *disjoint*.

A more general addition principle

How could we calculate $n(A \cup B)$ if A and B are not disjoint?

The earlier principle doesn't work: if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$, then $n(A) + n(B) = 10$, but $n(A \cup B) = 7$.

$n(A) + n(B)$ counts the overlapping elements (1, 3, and 5) twice!

We can fix this by subtracting the overcount back out again:

Generalized addition principle

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Example

Among 30 students, 13 play the piano, 16 play the guitar, and 5 play both instruments. How many students play at least one instrument? We might denote this with $n(A) = 13$, $n(B) = 16$, and $n(A \cap B) = 5$, so $n(A \cup B) = 13 + 16 - 5 = 24$.

A subtractive variant

For a universal set U, we know that $A \cup A' = U$.

In addition, $A \cap A' = \varnothing$.

Rearranging the additive principle $n(A) + n(A') - n(A \cap A') = n(U)$, we get:

Subtractive Principle

$$n(A') = n(U) - n(A)$$

Example

Among 30 students, 13 play the piano, 16 play the guitar, and 5 play both instruments. How many students play neither instrument?
In the last slide we saw that $n(A \cup B) = 24$; here $n(U) = 30$ so $n((A \cup B)') = 30 - 24 = 6$.

Using these rules in combination

Given the sizes of any four sets, we can usually work out the other sets of interest.

Finding an overlap

A cable company has 10000 subscribers. 3770 have cable internet, 3250 have digital phone, and 4530 have neither. How many subscribers have both services?
We could let U be the set of all subscribers, A those with internet, and B those with phone. So $n(U) = 10000$, $n(A) = 3770$, $n(B) = 3250$, and $n((A \cup B)') = 4530$. We want to find $n(A \cap B)$.

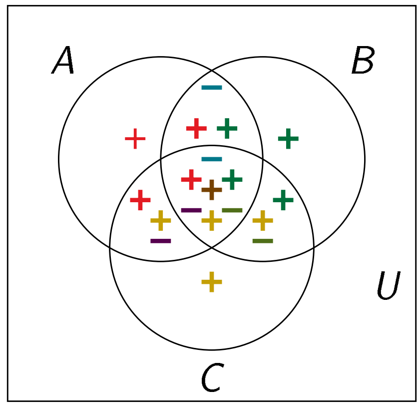
$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7020 - n(A \cap B)$$

$$n(A \cup B) = n(U) - n((A \cup B)') = 10000 - 4530 = 5470$$

so since $7020 - n(A \cap B) = 5470$, we know $n(A \cap B) = 1550$.

A more complicated additive principle

Using a Venn diagram, and trying to count every region *exactly once*, we can come up with a rule for finding the size of a union of three sets, but it's pretty messy!



$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

This result is also known as the *Inclusion-Exclusion Principle*.

A multiplicative principle

Often, we want to count the number of ways to make several choices in sequence.

Example

A license plate consists of any three letters followed by any three digits. How many distinct license plates are possible?

To answer a question like this, we need a principle for describing the number of ways to make *several choices in sequence*:

Multiplicative Principle

If we want to build an *ordered* list of an object from S_1, an object from S_2, and so forth up to S_r, the number of possible lists we can build is

$$n(S_1) \times n(S_2) \times n(S_3) \cdots \times n(S_r)$$

Application of the multiplicative principle

Example

A license plate consists of any three letters followed by any three digits. How many distinct license plates are possible?

In this case, our sets S_1, S_2, and S_3 are all $\{A, B, \ldots, Z\}$ and S_4, S_5, and S_6 are all $\{0, 1, 2, \ldots, 9\}$. We thus have

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

possible license plates.

A subtler application

A more complicated example

If a license plate consists of any three *different* letters followed by any three *different* digits, how many distinct license plates are possible?

Here it seems like S_2 depends on what we choose from S_1; whichever letter is chosen from S_1 shouldn't be in S_2.

The good news is which letter doesn't actually matter: no matter what, $n(S_2) = 25$ and $n(S_3) = 24$. Likewise, $n(S_5) = 9$ and $n(S_6) = 8$, so we get

$$26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000$$

possible license plates.

Combining rules

An additive/multiplicative example

A cafeteria will serve one of five plate lunches with any choice of 7 sides and either a salad or soup, or one of four sandwiches with a single side. How many different meal choices are possible?

We can get two different *types* of lunch. After calculating how many of each type are possible, we can add together the number of each type.

A plate lunch is a combination of: any of 5 mains, any of 7 sides, and any of 2 choices of salad/soup.
So there are $5 \times 7 \times 2 = 70$ plate lunches.

A sandwich is a combination of: any of 4 sandwiches, and any of 7 sides.
So there are $4 \times 7 = 28$ sandwich lunches.

We thus have $70 + 28 = 98$ lunches in total.

A complicated combination

Several rules in sequence

How many three-digit numbers have three different digits of the same parity (odd vs. even) and do not begin with a zero?

We can subdivide this problem into finding all the three-digit numbers with three different *odd* digits, and all the three-digit numbers with three different *even* digits (and no leading 0).
To get three odd digits, we have 5 choices for the first digit, 4 for the second, and 3 for the third: $5 \times 4 \times 3 = 60$ in total.
To get three even digits, we have 5 choices for the first digit, 4 for the second, and 3 for the third: $5 \times 4 \times 3 = 60$ in total, but this includes some with leading zeroes!
We *subtract* out those numbers starting with a zero and followed by two other even digits. We have 4 for the second digit and 3 for the third: $4 \times 3 = 12$.
So our total will be $60 + (60 - 12) = 108$.