Permutations and Combinations

MATH 107: Finite Mathematics

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Building strings without repetition

A familiar question

How many ways are there to build a string of four letters from $\{A, B, C, D, E, F\}$ if no letter can be used twice?

This is an easy question to answer with multiplication.

- Any letter can be first, so you have 6 choices.
- Any letter *except the one already used* can be second, so you have 5 choices.
- Any letter *except the two already used* can be third, so you have 4 choices.
- Any letter *except the three already used* can be fourth, so you have 3 choices.

Thus we can build any of $6 \times 5 \times 4 \times 3 = 360$ different strings.

How to generalize?

The previous question is one of a large family of variants: we could have any alphabet and any string length.

Definition

A *permutation of length k with n letters* is a string of length k made from those letters, using no letter more than once.

For instance, the 360 strings enumerated above were the *permutations of length 4 with 6 letters*.

Question

How many permutations of length k with n letters are there?

The general formula

Question

How many permutations of length k with n letters are there?

Let's consider a multiplicative approach:

- There are n choices for the first letter,
- $n-1$ choices for the second letter,
- $n-2$ choices for the third letter,
- and so forth up to $n-k+1$ choices for the kth letter.

so we can make this sequence of different choices in a total of

$$n(n-1)(n-2)(n-3)\cdots(n-k+1)$$

different ways.

Factorials

Counting permutations

There are

$$n(n-1)(n-2)\cdots(n-k+1)$$

permutations of length k with n letters.

We can introduce a new notation to simplify this product. Let *the factorial of n* be

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Then our count of permutations is

$$\frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots(3)(2)(1)}{(n-k)(n-k-1)\cdots(3)(2)(1)} = \frac{n!}{(n-k)!}$$

Calculation with Factorials

The small factorials are easily calculated:
$0! = 1$
$1! = 1 = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$
So, for instance, our original permutation question could have been solved with

$$\frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{720}{2} = 360$$

Ratios of large factorials

Factorials of even small numbers can be very large. For instance,

$$13! = 6,227,020,800$$

Thus, it might be impractical to calculate the ratio $\frac{40!}{35!}$ by determining the numerator and denominator.

Instead, we expand both and cancel common terms:

$$\frac{40!}{35!} = \frac{40 \times 39 \times 38 \times 37 \times 36 \times \cancel{35 \times \cdots \times 2 \times 1}}{\cancel{35 \times \cdots \times 2 \times 1}} = 40 \times 39 \times 38 \times 37 \times 36$$

which can be calculated to be 78,960,960.

The Permutation Statistic

Because counting permutations is useful, we denote a special sympol for it.

Definition

The *permutation statistic* $P_{n,k}$ is equal to

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)(n-3)\cdots(n-k+1)$$

For example, if I had five different gifts, and I wanted to give then to three different people, I could do so in $P_{5,3} = 20$ ways.

A useful application

How many ways are there to put five (distinguishable) people in a line?

There are 5 objects, and we're building an ordered list of length 5 with no repetitions, so it's $P_{5,5} = \frac{5!}{0!} = 120$.

Ignoring order

So far we've looked at selecting objects when *order matters*. How could we consider selecting objects when order *doesn't* matter?

Example question

How many ways are there to choose a 3-element subset of $\{1,2,3,4,5\}$?

{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}
{1,4,5}	{2,3,4}	{2,3,5}	{2,4,5}	{3,4,5}

There are 10, but how could we compute that?

These structures we know as *combinations*: like permutations, but without order.

An organizational scheme

There are 60 *permutations* of length 3 from an alphabet of size 5; let's group those.

123	124	125	134	135	145	234	235	245	345
132	142	152	143	153	154	243	253	254	354
213	214	215	314	315	415	324	325	425	435
231	241	251	341	351	451	342	352	452	453
312	412	512	413	513	514	423	523	524	534
321	421	521	431	531	541	432	532	542	543

Note that the 10 columns correspond to the *combinations* of length 3 from an alphabet of size 5, while the 6 rows correspond to the *orderings* of a specific combination.

Abstracting this approach

We have two different ways to count permutations.

We know that the number of permutations of length k from n objects is $P_{n,k}$.

But we could *also* build such a permutation by selecting a combination of length k from n objects (from some yet-unknown number of possibilities) and then ordering these k objects (in any of $k!$ ways).

Thus, if we denote the number of combinations by $C_{n,k}$, we have:

$$P_{n,k} = C_{n,k}k!$$

or

$$C_{n,k} = \frac{P_{n,k}}{k!} = \frac{n!}{(n-k)!k!}$$

Examples of the combination statistic

Choosing a committee

How many different ways could a 3-person committee be chosen from a 7-person group? $C_{7,3} = \frac{7!}{4!3!} = \frac{7\times6\times5}{3\times2\times1} = 35$ ways.

Choosing a meal

A plate lunch consists of any of 5 entrees, together with a choice of 2 out of 6 sides. How many plate lunches are possible? $5 \times C_{6,2} = 5 \times \frac{6!}{4!2!} = 5 \times \frac{6\times5}{2\times1} = 75$.

Building a poker hand

There are 52 cards in a deck and the order of the five cards in a draw poker hand is irrelevant. How many possible hands are there? $C_{52,5} = \frac{52!}{47!5!} = \frac{52\times51\times50\times49\times48}{5\times4\times3\times2\times1} = 2,598,960$

More fun with poker hands

Drawing five cards from a 52-card deck (4 suits, 13 numbers) is instructive. We can count *many* different types of poker hands.

Counting full houses

A full house is a collection of five cards with three of the same number and two more of a different identical number. How many ways are there to build a full house?

Construction process

An example like 3♥3♠3♣8♥8♦ is the result of several decisions:

- a *number* for the triplet (here, 3): 13 choices.
- a *different number* for the pair (here, 8): 12 choices.
- three *suits* for the triplet (here, ♥, ♠, and ♣): $C_{4,3}$ choices.
- two *suits* for the pair (here, ♥ and ♦): $C_{4,2}$ choices.

Thus there are $13 \times 12 \times C_{4,3} \times C_{4,2} = 3744$ different full houses.

Some other poker hands

Here's a list of several different types of poker hands, and the counts of each; you might want to try to figure out where these counts come from!

- Royal flush (AKQJT of a single suit): 4.
- Straight flush (5 in a row of a single suit, not royal): 4×9.
- Four of a kind: $13 \times C_{4,1} \times 12 \times C_{4,4}$.
- Flush (all same suit, not a straight): $4 \times (C_{13,5} - 10)$.
- Straight (5 in a row, not all the same suit): $4^4 \times 3 \times 10$.
- Three of a kind: $13 \times C_{4,3} \times C_{12,2} \times 4^2$.
- Two pair: $C_{13,2} \times C_{4,2} \times C_{4,2} \times 11 \times 4$.
- One pair: $13 \times C_{4,2} \times C_{12,3} \times 4^3$.

Summary:The important statistics

We can count the number of ways to draw k objects from a set of size n in four different ways, depending on the "rules" of our drawing:

- Repetitions allowed, order matters: $n \times n \times \cdots \times n = n^k$.
- Repetitions forbidden, order matters: $n(n-1)\cdots(n-k-1) = \frac{n!}{(n-k)!} = P_{n,k}$.
- Repetitions forbidden, order irrelevant: $\frac{n(n-1)\cdots(n-k-1)}{k(k-1)\cdots(1)} = \frac{n!}{(n-k)!k!} = C_{n,k}$.
- Repetitions allowed, order irrelevant: We aren't using it, but it's actually $C_{n+k-1,k-1}$.