Introductory Probability

MATH 107: Finite Mathematics

University of Louisville

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Probability in our daily lives

We see "chances", "odds", and "probabilities" everywhere:

- A weather report may give a "30% chance" of snow.
- The Kentucky Lottery gives "1:648976 odds" of a Match 4 Powerball win.
- A fair coin has a "1 in 2" chance of coming up heads.

What do all these numbers mean? And how do we find them?

Experiments, Sample Spaces and Events

An *experiment* is any random procedure, e.g.:

- spinning a roulette wheel,
- rolling a pair of 6-sided dice, or
- a Lotto drawing.

The *sample space* is the set of all individual possible results of the procedure, e.g.:

- for roulette: $\{0, 00, 1, 2, \ldots, 36\}$.
- for rolling two dice: $\{(1,1),(1,2),(1,3),\ldots,(6,6)\}$.
- for Powerball: a set with 175,223,510 elements.

An *event* is any set of conditions on the result of the procedure, e.g.:

- for roulette: the number 14 is shown.
- for rolling two dice: their sum is 9 or more.
- for Powerball: the drawing includes the numbers 6 and 20.

Probability is all about determining how likely certain events are to happen.

Describing events

Some events occur on single elements of the sample space. These are called *simple events*.

For example, the event "a roulette spin ends on 14" is simple.

Many events are complicated and consist of several elements of the sample space.

Here are the elements of the sample space for a roll of 2 dice:

(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

The red elements are the elements of the specific event "the dice show a sum of 9 or more".

A sample space is essentially a universal set, and events are *subsets* of it; simple events are single-element subsets.

Sample spaces and counting

The techniques learned in Chapter 7 are vital to counting and identifying the elements of a sample space.

For instance, if the experiment is drawing 5 cards from a 52-card deck, we know the sample space contains $C_{52,5}$ elements.

If the experiment is flipping a coin 6 times in sequence, the sample space contains 2^6 elements.

If the experiment is the random selection of recipients for 3 different prizes from among 20 people, the sample space contains $P_{20,3}$ elements.

Probability

Identifying sample spaces and their elements doesn't tell us how likely things are, though. For that we need a *probability function*, which assigns a number from 0 to 1 to every event.

Probability 0 (or 0%) is assigned to things which *never happen*.

Probability 1 (or 100%) is assigned to things which *always happen*.

Most interesting events are somewhere in between in probability. You intuitively know some probabilities:

- If a fair coin is flipped, the event "it shows heads" has probability 0.5.
- If a fair die is rolled, the event "4 is rolled" has probability $\frac{1}{6}$.

There are several fairly intuitive rules governing how probabilities can work.

What makes a probability function?

Definition

A *probability function* P on a sample space $S = \{e_1, e_2, \ldots, e_n\}$ is any function such that:

- For each i, $0 \le P(e_i) \le 1$.
- $P(e_1) + P(e_2) + P(e_3) + \cdots + P(e_n) = 1$

For instance, in a simple single-die-rolling experiment, we would have sample space $S = \{1, 2, 3, 4, 5, 6\}$ and probability function

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6)$$

if the die is fair.

Where do probability functions come from?

In practice, they come from scientific study, repeating an experiment many times.

Mathematically, they often arise from assumptions: e.g. that a die, coin, or lottery is “fair”.

Frequently—but not always—every simple event in the sample space has equal probability.

Examples of uniform probability

- If we draw 5 cards from a shuffled deck, each of the $C_{52,5}$ simple events has probability $\frac{1}{C_{52,5}}$.
- If we roll 3 fair 20-sided dice, each of the 20^3 events occurs with probability $\frac{1}{20^3}$.

More complicated probabilities

Examples of nonuniform probability

- An unfair coin might have sample space $\{H, T\}$ with $P(H) = 0.8$ and $P(T) = 0.2$.
- If we roll 6-sided dice until the total exceeds 3, there are 41 possibilities, not all of which are of the same probability.

Fortunately, we usually don't have extremely complicated probability functions.

Probabilities of nonsimple events

We have seen probabilities for *simple* events. How do we calculate the probability of an event consisting of multiple elements of the sample space?

Rules for calculating probabilities of events

- $P(\varnothing) = 0$.
- For any event e_i, $P(\{e_i\}) = P(e_i)$.
- $P(\{e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_r}\}) = P(e_{i_1}) + P(e_{i_2}) + P(e_{i_3}) + \cdots + P(e_{i_r})$.
- If S is the whole sample space, $P(S) = 1$.

For example, you might ask: when you roll a fair 6-sided die, what is the likelihood of a prime number?

$$P(\{2, 3, 5\}) = P(2) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

Probabilities in uniform-likelihood regimes

As we have seen, it is often the case that each simple event has probability $\frac{1}{n(S)}$.
Then for any event E, $P(E)$ will be the sum of $n(E)$ copies of $\frac{1}{n(S)}$.

Rule for uniform likelihood

If every element of the sample space S is equally likely, then $P(E) = \frac{n(E)}{n(S)}$.

Examples in uniform porbability

- Since there are 3744 full houses and 2598960 equally likely five-card hands, the chance of drawing a full house in five cards is $\frac{3744}{2598960} \approx 0.144\%$.
- Since there are 36 equally likely results on a roll of 2 dice, and 10 of them have a total of 9 or more, the chance of rolling at least a 9 is $\frac{10}{36} \approx 27.7\%$.

An exercise: Powerball!

Keeping the Lotto honest

A Powerball player chooses five numbers from 1–59 and one "power" number from 1–35. Then five white balls and one Powerball are randomly drawn. The lotto claims a $\frac{1}{12245}$ likelihood of matching 3 white numbers and the powerball. Is this correct?

All plays are of equal likelihood, so let's suppose we played (1,2,3,4,5) with powerball number 1. The drawing of balls constitutes an *experiment*.

The *sample space* S is a set containing all possible draws.

$$S = C_{59,5} \times 35 = \frac{59 \times 58 \times 57 \times 56 \times 55}{120} \times 35 = 175223510$$

and each element of the sample space is equally likely with probability $\frac{1}{175223510}$.

Powerball continued

Keeping the Lotto honest

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We want to find the size of the *event* "we draw exactly three of the numbers from 1–5 on white balls, and draw a 1 powerball".
We select 3 white numbers from 1–5 in any of $C_{5,3}$ ways, and two white numbers from 6–59 in any of $C_{54,2}$ ways, and then the powerball 1 in exactly one way.Thus our event contains

$$n(E) = C_{5,3} \times C_{54,2} \times 1 = \frac{5 \times 4}{2 \times 1} \times \frac{54 \times 53}{2 \times 1} \times 1 = 14310$$

simple events. So $P(E) = \frac{n(E)}{n(S)} = \frac{14310}{175223510}$ which is close to (although not exactly equal to) $\frac{1}{12245}$.