Probability with Set Operations

MATH 107: Finite Mathematics

University of Louisville

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Complicated Probability, 17th century style

Antoine Gombaud, Chevalier de Méré, was fond of gambling and posed a problem:

A motivational question

Which is better to bet on: the chance of rolling at least one "1" in four rolls of a die, or the chance of rolling at least one "snake-eyes" in 24 rolls of a pair of dice?

de Méré thought they should both have the same probability, but his gaming records suggested otherwise!

To answer this question, we need more sophisticated probability tools.

Viewing Events as Sets

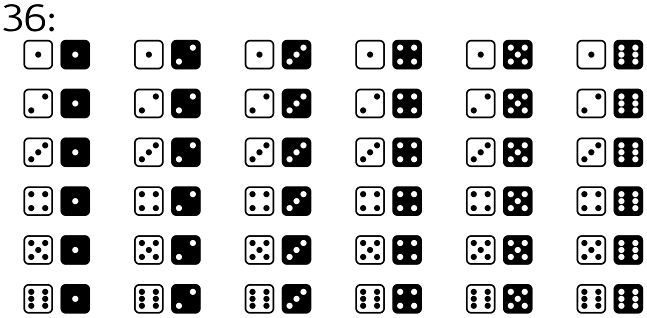
We previously saw that the *sample space* was a universal set, and that *events* are subsets of the sample space.

Because events are sets, we can perform the same operations that we do on sets.

We will see (possibly unsurprisingly) that such operations are fundamentally the same as *logic* operations.

An example: two dice

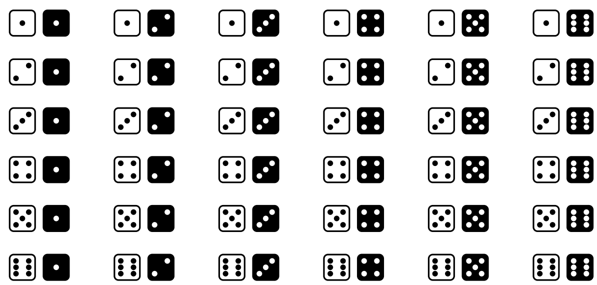
Suppose we roll two different dice; this is a sample space of size 36:



Let's describe some events on it!

- Let A be the event "the black die rolls a 3"; $P(A) = \frac{1}{6}$.
- Let B be the event "the white die rolls an even"; $P(B) = \frac{1}{2}$.
- Let C be the event "the sum of the dice is 4"; $P(C) = \frac{1}{12}$.
- Let D be the event "the sum of the dice is 8"; $P(D) = \frac{5}{36}$.

Explorations of operations



A is the event "the black die rolls a 3". B is the event "the white die rolls an even". C is the event "the sum of the dice is 4". D is the event "the sum of the dice is 8".

$P(A \cup B) = \frac{7}{12}$; $A \cup B$ is the event "the black die rolls 3 **or** the white die rolls even".

$P(A \cap B) = \frac{1}{12}$; $A \cap B$ is the event "the black die rolls 3 **and** the white die rolls even".

$P(A \cap C) = \frac{1}{36}$; $A \cap C$ is the event "The black die rolls 3 **and** the sum of the dice is 4".

$P(D') = \frac{31}{36}$; D' is the event "the sum of the dice is **not** 8".

Interpretation of event operations

In general, we thus see that whatever conditions exist on events, *set operations* on events correspond to *logical operations* on their conditions.

The event $A \cup B$ is the same as stating that "the conditions associated with A **or** B occur".

The event $A \cap B$ is the same as stating that "the conditions associated with A **and** B both occur".

The event A' is the same as stating that "the conditions associated with A do **not** occur".

Calculations with event operations

There are two simple formulas one can use to calculate many probabilities. Recall that $P(A)$ is conceptually similar to $n(A)$, and these will make sense:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A') = 1 - P(A)$$

With these on hand, some calculations can be easy:

$$P(\text{one roll of a die does not get a 5}) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$P(\text{two rolls of a die get at least one 5}) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$$

A clever trick with complements

Sometimes, it is *easier* to figure out the probability of an event's complement.

A difficult calculation

I roll a six-sided die 4 times. What is the probability that I roll the same number at least twice?

If we were to call the above event A, finding $P(A)$ directly is difficult, but $P(A')$ is surprisingly tractable!

A complementary event

I roll the die 4 times. What is the probability that I *do not* roll the same number at least twice; i.e. all my rolls are different?

This is easily calculated as the fraction $\frac{P_{6,4}}{6^4} = \frac{360}{1296} = \frac{5}{18}$.

So since $P(A') = \frac{5}{18}$, $P(A) = 1 - \frac{5}{18} = \frac{13}{18}$.

A classic question: the birthday paradox

The puzzle

Among n people, what is the probability that two of them have the same birthday? (Let's simplify by assuming every day is equally likely, and that there are no leap years)

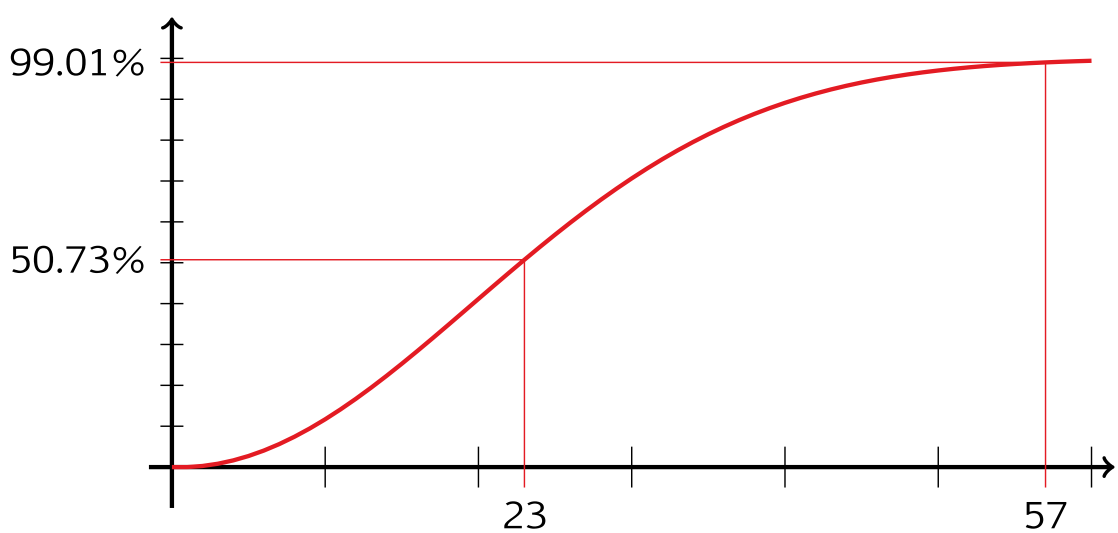
A simple observation: if $n = 1$, our probability is 0; if $n \geq 366$, our probability is 1. In between might be more interesting.

This is basically the same as our die-rolling problem, but with 365 possible results from each roll!

The complement event is that everyone has a different birthday, which can occur in $P_{365,n}$ ways out of 365^n possibilities in the sample space, so

$$P(A) = 1 - P(A') = 1 - \frac{P_{365,n}}{365^n}$$

Birthday paradox, continued



Surprisingly, as few as 23 people are more likely to have a shared birthday than not!

With only 57 people, a shared birthday becomes a near-certainty.

What about poor de Méré?

Remember this?

Which is better to bet on: the chance of rolling at least one "1" in four rolls of a die, or the chance of rolling at least one "snake-eyes" in 24 rolls of a pair of dice?

Let's call these events A and B.
Let's look at $P(A')$, the chance of *not* rolling any 1s in four rolls of the dice:

$$P(A') = \left(\frac{5}{6}\right)^4 \approx 48.22\%$$

and at $P(B')$, the chance of *not* rolling snake-eyes in 24 rolls:

$$P(B') = \left(\frac{35}{36}\right)^{24} \approx 50.85\%$$

Thus the original two bets have a 51.77% and 49.14% chance respectively.

Other ways of describing probability

We have thus far described probability in terms of its chance of happening.

Such descriptions are as fractions, decimals, or percentages.

Three ways to say the same thing

The likelihood of rolling a one on a single throw of a die is $\frac{1}{6}$, about 0.167 or 16.7%.

An alternative phrasing

When we roll a die, we will roll a one about $\frac{1}{5}$ as often as we roll something else.

Instead of determining how likely something is relative to all possibilities, we can determine how its likelihood compares to the likelihood of not happening. We call these *odds*.

Formulation of odds

The formal definition of odds

An event A with probability $P(A)$ has *odds for the event* of $\frac{P(A)}{P(A')} = \frac{P(A)}{1-P(A)}$. We often write odds not as the fraction $\frac{a}{b}$ but as the ratio $a : b$.

For example, the odds for a 6-sided die rolling 1 are $1:5$.

The odds for a coin landing heads-up are $1:1$.

The odds for a pair of dice rolling a sum greater than 5 is $13:5$ (note here $P(A) = \frac{26}{36} = \frac{13}{18}$).

Odds, continued

Odds for and against events

An event A with probability $P(A)$ has *odds for the event* of $\frac{P(A)}{P(A')} = \frac{P(A)}{1-P(A)}$.
An event A with probability $P(A)$ has *odds against the event* of $\frac{P(A')}{P(A)} = \frac{1-P(A)}{P(A)}$.

Since the odds for a 6-sided die rolling 1 are $1:5$, the odds against it at $5:1$.
You can interpret odds of $1:1$ as "equally likely to happen as not", and ratios larger or smaller than that are deviations from equality.
Odds answer the question: how do you bet to make a fair game? For instance, if you wanted a fair game dealing with rolling a single one on a die, the player betting against it should put up 5 times as much money as the other player.