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MATH 107: Finite Mathematics

University of Louisville

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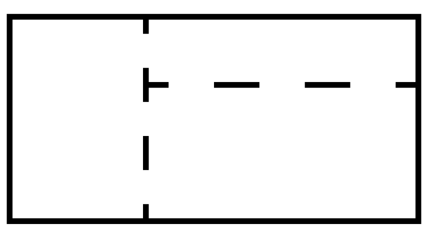
A Simple Question

Consider this simple maze:



Suppose a mouse is in the leftmost chamber, and every second, it chooses randomly to go through one of the holes or to stay where it is, with equal probability. Where is the mouse likely to be in 2 seconds? In 5 seconds? In 10 seconds?

This is an example of a repeated *state transition*.

Another transitional problem

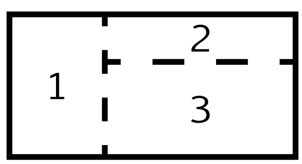
A medical scenario

A population is subject to a disease for which there's a somewhat ineffective vaccine. Each week, from the unvaccinated population, 20% catch the disease and 30% receive the vaccine. From among the vaccinated population, 5% of the population lose the vaccine and immediately get sick. Among the sick population, 50% recover per week.
If 10% of our population is originally sick and nobody is vaccinated, what will our population look like in 5 weeks?

This is a lot like our mouse-maze; instead of having 3 rooms, we have 3 possible states (sick, well, and vaccinated). Instead of having probabilities of change, we have proportions of change.

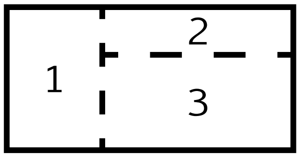
Thus, this too is an example of a repeated *state transition*.

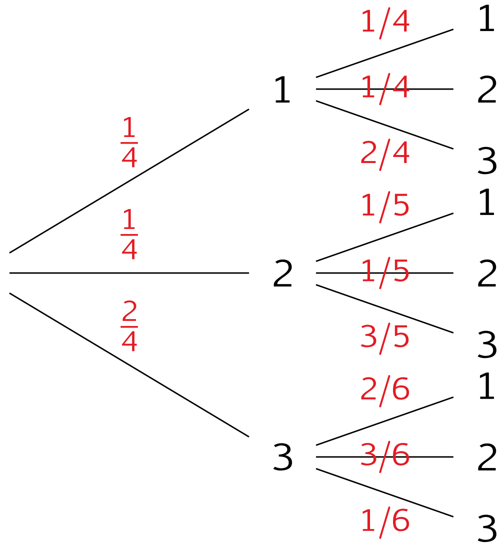
Formalizing the idea of state transition



The idea here is that we have probabilities (or proportions) associated with *initial states*, and then we *step through time* performing the rearrangements of probabilities described by the system.
For instance, our initial state is that the mouse is 100% likely to be in chamber 1.
After 1 second, it has a 25% chance of staying, a 25% chance of going to chamber 2, and a 50% chance of going to chamber 3.
After 2 seconds, it has about a 27.92% chance of being in chamber 1, a 36.25% chance of being in chamber 2, and a 34.83% chance of being in chamber 3.
How did I get that second set of probabilities? Read on!

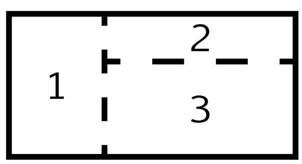
State transitions in a tree: 2 steps



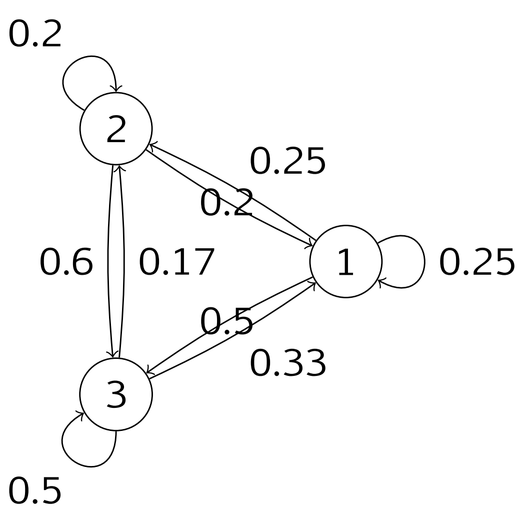


$$P(1) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{5} + \frac{2}{4} \times \frac{2}{6} = \frac{67}{240} \approx 27.92\%$$
$$P(2) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{5} + \frac{2}{4} \times \frac{3}{6} = \frac{29}{80} = 36.25\%$$
$$P(3) = \frac{1}{4} \times \frac{2}{4} + \frac{1}{4} \times \frac{3}{5} + \frac{2}{4} \times \frac{1}{6} = \frac{43}{120} \approx 35.83\%$$

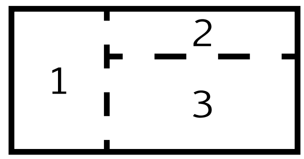
Visualizing state transition



There are two useful ways to visualize state transitions. One is with a *network*:



Visualizing state transition, cont'd

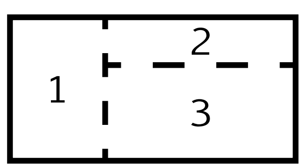


The other useful visualization is with a *transition matrix*, where rows represent entry states, and columns exit states.

$$\begin{bmatrix} 1/4 & 1/4 & 2/4 \\ 1/5 & 1/5 & 3/5 \\ 2/6 & 3/6 & 1/6 \end{bmatrix}$$

This particular representation lends itself to a useful computation!

Using transition matrices



$$P = \begin{bmatrix} 1/4 & 1/4 & 2/4 \\ 1/5 & 1/5 & 3/5 \\ 2/6 & 3/6 & 1/6 \end{bmatrix}$$

Our *initial state* is a 100% chance of being in chamber 1. We can represent that with the *initial state vector* $S_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Then, the mouse's probability distribution after one second is given by

$$S_1 = S_0 P = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

and after two seconds:

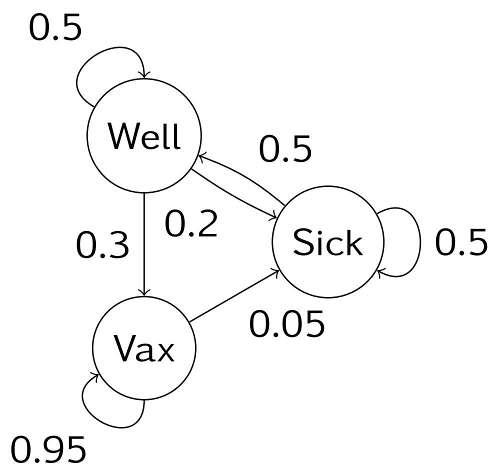
$$S_2 = S_1 P = \begin{bmatrix} \frac{67}{240} & \frac{29}{80} & \frac{43}{120} \end{bmatrix}$$

So there is a standard procedure for seeing how a probability of state transitions effects state over time.

The same approach to population modeling

The already-seen medical scenario

Each week, of the unvaccinated, 20% catch the disease and 30% receive the vaccine. Among the vaccinated population, 5% immediately get sick. Among the sick, 50% recover per week. 10% of our population is originally sick and nobody is vaccinated.



$$P = \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} \leftarrow \text{Vax} \\ \leftarrow \text{Well} \\ \leftarrow \text{Sick} \end{matrix}$$

$$S_0 = \begin{bmatrix} 0 & 0.9 & 0.1 \end{bmatrix}$$

Answering questions with powers of matrices

$$P = \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{matrix} \leftarrow \text{Vax} \\ \leftarrow \text{Well} \\ \leftarrow \text{Sick} \end{matrix}$$

$$S_0 = \begin{bmatrix} 0 & 0.9 & 0.1 \end{bmatrix}$$

In this scenario, we might ask what the population looks like in 5 weeks? In other words, what is S_5?

$$S_5 = S_4 P = S_3 P^2 = S_2 P^3 = S_1 P^4 = S_0 P^5$$

We could multiply the vector by the matrix 5 times, but there's an easier way!

$$P^2 = \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.9025 & 0.025 & 0.0725 \\ 0.435 & 0.35 & 0.215 \\ 0.15 & 0.5 & 0.35 \end{bmatrix}$$

Powers of matrices, cont'd

$$P^4 = P^2P^2 = \begin{bmatrix} 0.9025 & 0.025 & 0.025 \\ 0.435 & 0.35 & 0.215 \\ 0.15 & 0.5 & 0.35 \end{bmatrix} \begin{bmatrix} 0.9025 & 0.025 & 0.025 \\ 0.435 & 0.35 & 0.215 \\ 0.15 & 0.5 & 0.35 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.8363 & 0.0676 & 0.0962 \\ 0.5771 & 0.2409 & 0.1820 \\ 0.4054 & 0.3538 & 0.2409 \end{bmatrix}$$

$$P^5 = P^4P \approx \begin{bmatrix} 0.8363 & 0.0676 & 0.0962 \\ 0.5771 & 0.2409 & 0.1820 \\ 0.4054 & 0.3538 & 0.2409 \end{bmatrix} \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.8147 & 0.0819 & 0.1034 \\ 0.6205 & 0.2115 & 0.1680 \\ 0.4912 & 0.2973 & 0.2115 \end{bmatrix}$$

Powers of matrices, cont'd (2)

$$P^5 \approx \begin{bmatrix} 0.8147 & 0.0819 & 0.1034 \\ 0.6205 & 0.2115 & 0.1680 \\ 0.4912 & 0.2973 & 0.2115 \end{bmatrix}$$

so

$$S_5 = S_0P^5 \approx \begin{bmatrix} 0 & 0.9 & 0.1 \end{bmatrix} \begin{bmatrix} 0.8147 & 0.0819 & 0.1034 \\ 0.6205 & 0.2115 & 0.1680 \\ 0.4912 & 0.2973 & 0.2115 \end{bmatrix}$$

$$\approx \begin{bmatrix} 0.6075 & 0.2201 & 0.1724 \end{bmatrix}$$

Thus, after 5 weeks, 60.75% of the population will be vaccinated, 22.01% well, and 17.24% sick.

Some Properties of Transition Matrices

- The *ij*th entry represents the probability (or proportion) of things in state *i* transitioning to state *j*.
- Every entry is non-negative.
- Every row adds up to 1.

The following could not be a transition matrix, for any system:

$$\begin{bmatrix} 0 & 0 & 0.5 & 0.25 \\ 0.1 & 0.2 & 0.3 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

The last three rows are good, but the first row has total 0.75!