Regular Markov Chains

MATH 107: Finite Mathematics

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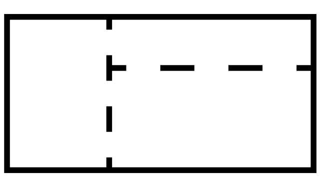
Long-term behaviors



$$P = \begin{bmatrix} 1/4 & 1/4 & 2/4 \\ 1/5 & 1/5 & 3/5 \\ 2/6 & 3/6 & 1/6 \end{bmatrix}; S_0 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Remember this maze from the last lesson; we can use computers to find long-term behavior:

$$S_2 = SP^2 \approx \begin{bmatrix} 0.27917 & 0.36250 & 0.35833 \end{bmatrix}$$
$$S_5 = SP^5 \approx \begin{bmatrix} 0.26586 & 0.33140 & 0.40274 \end{bmatrix}$$
$$S_{10} = SP^{10} \approx \begin{bmatrix} 0.26668 & 0.33335 & 0.39997 \end{bmatrix}$$
$$S_{20} = SP^{20} \approx \begin{bmatrix} 0.26667 & 0.33333 & 0.40000 \end{bmatrix}$$
$$S_{21} = SP^{21} \approx \begin{bmatrix} 0.26667 & 0.33333 & 0.40000 \end{bmatrix}$$

We might conjecture that this system is approaching a *stationary state* looking a lot like $\begin{bmatrix} \frac{4}{15} & \frac{1}{3} & \frac{2}{5} \end{bmatrix}$

More long-term behavior

$$\begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0.0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}; S_0 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix}$$

Alas, not every transition matrix reaches a stationary state!

$$S_1 = SP = \begin{bmatrix} 0.09 & 0.70 & 0.21 \end{bmatrix}$$
$$S_2 = SP^2 = \begin{bmatrix} 0.21 & 0.30 & 0.49 \end{bmatrix}$$
$$S_3 = SP^3 = \begin{bmatrix} 0.09 & 0.70 & 0.21 \end{bmatrix}$$

and the system oscillates between these two states forever!

Key questions

- When does a system have a unique stationary state?
- What is that stationary state?

Stationary states

Some matrices, like $\begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0.0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$, have no stationary state.

Some, like $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.3 & 0.7 \end{bmatrix}$ have many stationary states.

We want to know which transition matrices have exactly one stationary state.

The regularity criterion

Definition

A matrix P is *regular* if there is some number n such that every entry of P^n is nonzero.
A Markov chain is *regular* if its transition matrix is regular.

So, for instance, our maze example $\begin{bmatrix} 1/4 & 1/4 & 2/4 \\ 1/5 & 1/5 & 3/5 \\ 2/6 & 3/6 & 1/6 \end{bmatrix}$ is regular, and the matrices $\begin{bmatrix} 0 & 1 & 0 \\ 0.3 & 0.0 & 0.7 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.3 & 0.7 \end{bmatrix}$ are not.

Theorem

A Markov chain has a unique stationary state if and only if it is regular.

We won't prove that here; details are in MATH 325.

Testing regularity

To determine whether a matrix is regular, we look at powers of it until either we have all nonzero entries or we repeat entries.

Fortunately, in doing this test, we only need consider whether entries are zero or nonzero, not their arithmetic values.

Example

Is $P = \begin{bmatrix} 0 & 0.3 & 0.7 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{bmatrix}$ regular?

$P = \begin{bmatrix} 0 & \sim & \sim \\ \sim & 0 & \sim \\ \sim & 0 & 0 \end{bmatrix}$; $P^2 = \begin{bmatrix} \sim & 0 & \sim \\ \sim & \sim & \sim \\ 0 & \sim & \sim \end{bmatrix}$; $P^3 = \begin{bmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & 0 & \sim \end{bmatrix}$; $P^4 = \begin{bmatrix} \sim & \sim & \sim \\ \sim & \sim & \sim \\ \sim & \sim & \sim \end{bmatrix}$, so P is regular!

Finding stationary vectors

Having determined when a matrix has a stationary state, we now want to find that state.

What we want is a vector S such that $SP = S$.

Also, since S describes a probability or proportion, we need its entries to add up to 1.

$$SP = S \Rightarrow SP = SI_n \Rightarrow SP - SI_n = \mathbf{0} \Rightarrow S(P - I_n) = 0$$

and in addition the entries of S add up to 1.

If we name the entries of S, we can thus solve the associated system.

Case study: mouse in a maze

$$P = \begin{bmatrix} 1/4 & 1/4 & 2/4 \\ 1/5 & 1/5 & 3/5 \\ 2/6 & 3/6 & 1/6 \end{bmatrix}$$

This is our "mouse-in-a-maze" transition matrix. Let's find a stationary $S = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$.

$$S(P - I_3) = \mathbf{0} \text{ and } s_1 + s_2 + s_3 = 1$$

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} -3/4 & 1/4 & 1/2 \\ 1/5 & -4/5 & 3/5 \\ 1/3 & 1/2 & -5/6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} -\frac{3}{4}s_1 + \frac{1}{5}s_2 + \frac{1}{3}s_3 = 0 \\ \frac{1}{4}s_1 - \frac{4}{5}s_2 + \frac{1}{2}s_3 = 0 \\ \frac{1}{2}s_1 + \frac{3}{5}s_2 - \frac{5}{6}s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases}$$

$$s_1 + s_2 + s_3 = 1$$

Case study: mouse in a maze (cont'd)

$$\begin{cases} -\frac{3}{4}s_1 + \frac{1}{5}s_2 + \frac{1}{3}s_3 = 0 \\ \frac{1}{4}s_1 - \frac{4}{5}s_2 + \frac{1}{2}s_3 = 0 \\ \frac{1}{2}s_1 + \frac{3}{5}s_2 - \frac{5}{6}s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases} \Longrightarrow \begin{cases} 45s_1 - 12s_2 - 20s_3 = 0 \\ 5s_1 - 16s_2 + 10s_3 = 0 \\ 15s_1 + 18s_2 - 25s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases}$$

which we can solve by reducing $\left[\begin{array}{ccc|c} 45 & -12 & -20 & 0 \\ 5 & -16 & 10 & 0 \\ 15 & 18 & -25 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 45 & -12 & -20 & 0 \\ 5 & -16 & 10 & 0 \\ 15 & 18 & -25 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{4}{15} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which is the result we expected way back when!

Case study: ineffective vaccine

$$P = \begin{bmatrix} 0.95 & 0 & 0.05 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

This is our vaccination transition matrix from the last lecture. Let's find a stationary $S = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$.

$$\begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix} \begin{bmatrix} -0.05 & 0 & 0.05 \\ 0.3 & -0.5 & 0.2 \\ 0 & 0.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$s_1 + s_2 + s_3 = 1$$

$$\begin{cases} -0.05s_1 + 0.3s_2 = 0 \\ 0 \quad s_1 - 0.5s_2 + 0.5s_3 = 0 \\ 0.05s_1 + 0.2s_2 - 0.5s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases}$$

Case study: ineffective vaccine (cont'd)

$$\begin{cases} -0.05s_1 + 0.3s_2 = 0 \\ -0.5s_2 + 0.5s_3 = 0 \\ 0.05s_1 + 0.2s_2 - 0.5s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases} \implies \begin{cases} 5s_1 - 30s_2 = 0 \\ s_2 - s_3 = 0 \\ 5s_1 + 20s_2 - 50s_3 = 0 \\ s_1 + s_2 + s_3 = 1 \end{cases}$$

which we can solve by reducing $\left[\begin{array}{ccc|c} 5 & -30 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 5 & 20 & -50 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 5 & -30 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 5 & 20 & -50 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{8} \\ 0 & 0 & 1 & \frac{1}{8} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so 75% of the population will end up vaccinated long-term, while the remaining 25% are evenly split between being well and ill.

A more typical example: 2-state systems

Mercifully, you will usually be spared 3-state systems. More usually you will be given a 2-state system.

Commuter preferences

Every month, 20% of rapid transit users switch to driving, while 30% of car drivers switch to rapid transit. What is the long-term proportion of drivers to transit users?

Here, $P = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{matrix} \leftarrow \text{RT} \\ \leftarrow \text{Car} \end{matrix}$.

And we want a solution to $\begin{bmatrix} s_1 & s_2 \end{bmatrix} \begin{bmatrix} -0.2 & 0.2 \\ 0.3 & -0.3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

with $s_1 + s_2 = 1$. Thus, we have the system

$$\begin{cases} -0.2s_1 + 0.3s_2 = 0 \\ 0.2s_1 - 0.3s_2 = 0 \\ s_1 + s_2 = 1 \end{cases}$$

2-state systems (cont'd)

$$\begin{cases} -0.2s_1 + 0.3s_2 = 0 \\ 0.2s_1 - 0.3s_2 = 0 \\ s_1 + \quad s_2 = 1 \end{cases} \Rightarrow \begin{cases} 2s_1 - 3s_2 = 0 \\ s_1 + \quad s_2 = 1 \end{cases}$$

We could use, for instance, substitution to solve this: $s_1 = \frac{3}{2}s_2$, so:

$$\frac{3}{2}s_2 + s_2 = 1 \Rightarrow \frac{5}{2}s_2 = 1 \Rightarrow s_2 = \frac{2}{5}$$

and then $s_1 = \frac{3}{2} \times \frac{2}{5} = \frac{3}{5}$.

Thus 60% of the population would use rapid transit and 40% cars.