Absorbing Markov Chains

MATH 107: Finite Mathematics

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Inescapable states

Regular Markov chains are those where every state is, eventually, reachable from every other.

However, many real-world situations have states which, once reached, don't change.

Examples of inescapable states

- In an epidemic model: vaccinated and dead
- In an mouse maze: a chamber with food
- In a consumer preference system: a long-term contract

Regular Markov chains will *not* model these, because those states are never left!

Absorbing chains and absorbing states

A state of a Markov chain is called *absorbing* if it is one which never transitions to another state.

A Markov chain where it is possible (perhaps in several steps) to get from every state to an absorbing state is called a *absorbing Markov chain*.

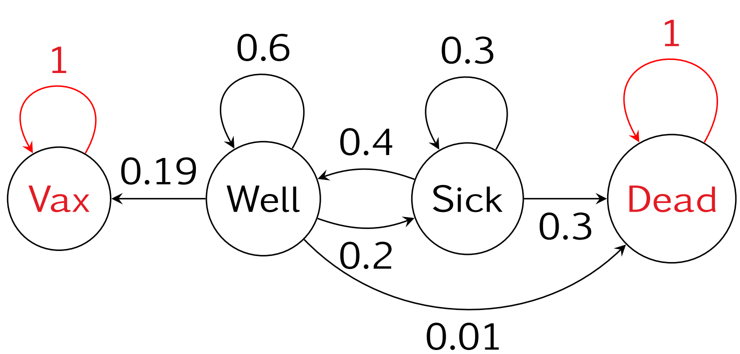
The situations described on the last slide are well modeled by absorbing Markov chains.

An epidemic-modeling Markov chain

Disease spreading

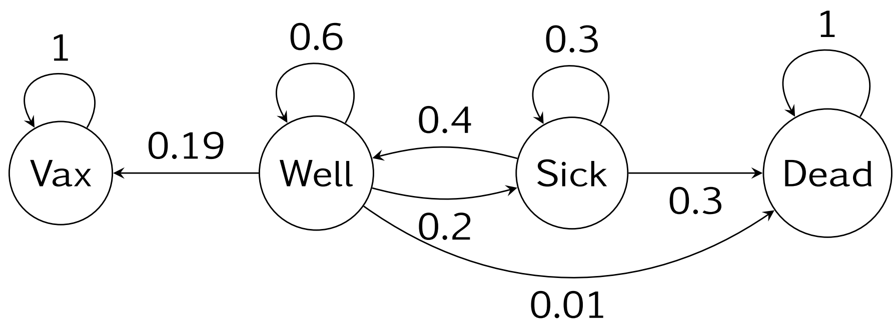
A virulent and deadly but vaccinatable disease is raging in a population. Every day, 40% of the sick recover and 30% of the sick die, while from the healthy, 1% die, 19% are vaccinated, and 20% get sick. What happens over the long term?

We might begin with a state diagram.



Note that the absorbing states have characteristic outflows!

Epidemic-modeling with matrices



We could put this same state data into a matrix:

$$P = \begin{array}{c} \begin{matrix} \text{Dead} & \text{Sick} & \text{Well} & \text{Vax} \end{matrix} \\ \begin{bmatrix} \color{red}{1.00} & \color{red}{0.00} & \color{red}{0.00} & \color{red}{0.00} \\ 0.30 & 0.30 & 0.40 & 0.00 \\ 0.01 & 0.20 & 0.60 & 0.19 \\ \color{red}{0.00} & \color{red}{0.00} & \color{red}{0.00} & \color{red}{1.00} \end{bmatrix} \end{array} \begin{array}{l} \color{red}{\text{Dead}} \\ \text{Sick} \\ \text{Well} \\ \color{red}{\text{Vax}} \end{array}$$

Note that absorbing states have characteristic associated rows!

Identifying absorbing states

We thus have three ways of identifying absorbing states.

- In a description, look for "naturally inescapable" states.
- In a state diagram, look for states with a self-directed arrow of weight 1.
- In a matrix, look for rows with a "1" on the diagonal entry.

Once we have identified absorbing states, we can be certain that long-term, repeated application of the matrix will put everything into one of these states.

Application of an absorbing Markov chain

$$P = \begin{array}{c} \begin{matrix} \text{Dead} & \text{Sick} & \text{Well} & \text{Vax} \end{matrix} \\ \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.30 & 0.30 & 0.40 & 0.00 \\ 0.01 & 0.20 & 0.60 & 0.19 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{matrix} \text{Dead} \\ \text{Sick} \\ \text{Well} \\ \text{Vax} \end{matrix} \end{array}$$

Let us consider the effect of this chain on a population where 10% were originally sick and 90% well.

$$S_0 = \begin{bmatrix} 0 & 0.1 & 0.9 & 0 \end{bmatrix}$$
$$S_1 = S_0 P = \begin{bmatrix} 0.039 & 0.21 & 0.58 & 0.171 \end{bmatrix}$$
$$S_5 = S_0 P^5 \approx \begin{bmatrix} 0.246 & 0.083 & 0.199 & 0.474 \end{bmatrix}$$
$$S_{20} = S_0 P^{20} \approx \begin{bmatrix} 0.361 & 0.002 & 0.004 & 0.633 \end{bmatrix}$$
$$S_{99} = S_0 P^{99} \approx \begin{bmatrix} 0.3635 & 0.000 & 0.000 & 0.6365 \end{bmatrix}$$

so, long-term, this epidemic ends with 36.35% of the population dead and 63.65% vaccinated.

Non-regularity of absorbing Markov chains

$$P = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.30 & 0.30 & 0.40 & 0.00 \\ 0.01 & 0.20 & 0.60 & 0.19 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{matrix} \text{Dead} \\ \text{Sick} \\ \text{Well} \\ \text{Vax} \end{matrix}$$

Recall, these chains are not regular—a different start scenario has a different outcome. Suppose 50% were originally sick and 50% well.

$$S_0 = \begin{bmatrix} 0 & 0.5 & 0.5 & 0 \end{bmatrix}$$
$$S_1 = S_0 P = \begin{bmatrix} 0.155 & 0.25 & 0.5 & 0.095 \end{bmatrix}$$
$$S_5 = S_0 P^5 \approx \begin{bmatrix} 0.367 & 0.078 & 0.184 & 0.370 \end{bmatrix}$$
$$S_{20} = S_0 P^{20} \approx \begin{bmatrix} 0.475 & 0.001 & 0.004 & 0.519 \end{bmatrix}$$
$$S_{99} = S_0 P^{99} \approx \begin{bmatrix} 0.4775 & 0.000 & 0.000 & 0.5225 \end{bmatrix}$$

so in this scenario, 47.75% die and 52.25% are vaccinated.

Characterizing long-term behavior

To discover the long-term behavior of this non-regular chain, we end up needing to find the *limiting matrix*.

Definition

The *limiting matrix* $\overline{P}$ of a transition matrix P is the matrix which P^n tends towards as n gets very large.

One important property of the limiting matrix is that $\overline{P}P = \overline{P}$; much like the stationary vector, it is unaffected by multiplication by P.

We could calculate $\overline{P}$ by brute force, finding a very large power of P, but there are better ways!

Reordering states and standard form

The order of states in a matrix is arbitrary, but we will describe a matrix as being in *standard form* if absorbing states precede the other states.

$$\begin{array}{c} \begin{array}{cccc} \text{Dead} & \text{Sick} & \text{Well} & \text{Vax} \end{array} \\ \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.30 & 0.30 & 0.40 & 0.00 \\ 0.01 & 0.20 & 0.60 & 0.19 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \begin{array}{c} \text{Dead} \\ \text{Sick} \\ \text{Well} \\ \text{Vax} \end{array} \end{array} \Rightarrow \begin{array}{c} \begin{array}{cccc} \text{Dead} & \text{Vax} & \text{Well} & \text{Sick} \end{array} \\ \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.01 & 0.19 & 0.60 & 0.20 \\ 0.30 & 0.00 & 0.40 & 0.30 \end{bmatrix} \begin{array}{c} \text{Dead} \\ \text{Vax} \\ \text{Well} \\ \text{Sick} \end{array} \end{array}$$

For instance, in our epidemic model, we'd like to reorder the entries to get vaccinated and dead at the beginning (in some order)

Standard form structure

$$P = \left[\begin{array}{cc|cc} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ \hline 0.01 & 0.19 & 0.60 & 0.20 \\ 0.30 & 0.00 & 0.40 & 0.30 \end{array}\right]$$

Standard form matrices always have a particular structure:

$$P = \left[\begin{array}{c|c} I & 0 \\ \hline Q & R \end{array}\right]$$

Here, for instance, $Q = \begin{bmatrix} 0.01 & 0.19 \\ 0.30 & 0.00 \end{bmatrix}$ and $R = \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}$.

In addition, we would expect $\overline{P} = \left[\begin{array}{c|c} I & 0 \\ \hline S & 0 \end{array}\right]$. All we need is S!

A spoiler and some justification

Theorem

If $P = \left[\begin{array}{c|c} I & 0 \\ \hline Q & R \end{array}\right]$, *then* $\overline{P} = \left[\begin{array}{c|c} I & 0 \\ \hline (I-R)^{-1}Q & 0 \end{array}\right]$.

$$\left[\begin{array}{c|c} I & 0 \\ \hline S & 0 \end{array}\right] = \overline{P} = P\overline{P} = \left[\begin{array}{c|c} I & 0 \\ \hline Q & R \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline S & 0 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline Q+RS & 0 \end{array}\right]$$

so we want $S = Q + RS$. Thus $(I-R)S = Q$ and $S = (I-R)^{-1}Q$.

Long-term behavior in our epidemic model

$$P = \left[\begin{array}{cc|cc} 1.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ \hline 0.01 & 0.19 & 0.60 & 0.20 \\ 0.30 & 0.00 & 0.40 & 0.30 \end{array}\right]$$

The relevant corner of $\overline{P}$ is going to be

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.3 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0.01 & 0.19 \\ 0.30 & 0.00 \end{bmatrix}$$

so as our first step we'd calculate the inverse of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.6 & 0.2 \\ 0.4 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ -0.4 & 0.7 \end{bmatrix}$$

Bet you hoped we were done with inverses!

We can find $\begin{bmatrix} 0.4 & -0.2 \\ -0.4 & 0.7 \end{bmatrix}^{-1}$ using the known 2×2 inverse formula or Gauss-Jordan elimination:

$$\left[\begin{array}{cc|cc} 0.4 & -0.2 & 1 & 0 \\ -0.4 & 0.7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -0.5 & 2.5 & 0 \\ -0.4 & 0.7 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & -0.5 & 2.5 & 0 \\ 0 & 0.5 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & -0.5 & 2.5 & 0 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 3.5 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right]$$

so $\begin{bmatrix} 0.4 & -0.2 \\ -0.4 & 0.7 \end{bmatrix}^{-1} = \begin{bmatrix} 3.5 & 1 \\ 2 & 2 \end{bmatrix}$.

The limiting matrix, at last

Since we have $(I-R)^{-1}$, we only need a multiplication to find $(I-R)^{-1}Q$. So

$$S = \begin{bmatrix} 3.5 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0.01 & 0.19 \\ 0.30 & 0.00 \end{bmatrix} = \begin{bmatrix} 0.335 & 0.665 \\ 0.620 & 0.380 \end{bmatrix}$$

and thus

$$\overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.335 & 0.665 & 0 & 0 \\ 0.620 & 0.380 & 0 & 0 \end{bmatrix}$$

From our original question

Recall our computational results:

Mortality rates

10% initially sick: 36.35% dead at end
50% initially sick: 47.75% dead at end

Does our limiting matrix $\overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.335 & 0.665 & 0 & 0 \\ 0.620 & 0.380 & 0 & 0 \end{bmatrix}$ bear this out?

$$\begin{bmatrix} 0 & 0 & 0.9 & 0.1 \end{bmatrix} \overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.335 & 0.665 & 0 & 0 \\ 0.620 & 0.380 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.3635 & 0.6365 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0.5 & 0.5 \end{bmatrix} \overline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.335 & 0.665 & 0 & 0 \\ 0.620 & 0.380 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.4775 & 0.5225 & 0 & 0 \end{bmatrix}$$