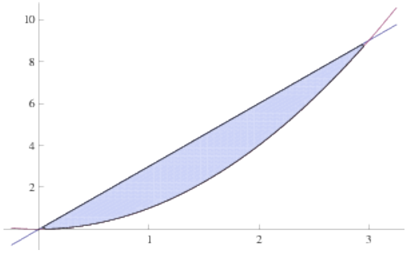
1. **(15 points)** *The region shown below is the area between the curves $y = 3x$ and $y = x^2$. Find the center of mass of this region.*



We must calculate the area, x-moment, and y-moment to find the center of mass.

The region is bounded on the left by $x = 0$ and on the right by $x = 3$. The upper curve is $y = 3x$ and the lower curve is $y = x^2$. Thus, the integral to calculate the area is

$$A = \int_0^3 3x - x^2 dx = \frac{3x^2}{2} - \frac{x^3}{3}\Big]_0^3 = \left(\frac{27}{2} - \frac{27}{3}\right) - (0 - 0) = \frac{9}{2}$$

To calculate the x=moment, we multiply the integrand above by x to get

$$M_x = \int_0^3 3x^2 - x^3 dx = x^3 - \frac{x^4}{4}\Big]_0^3 = \left(27 - \frac{81}{4}\right) - (0 - 0) = \frac{27}{4}$$

To calculate the y-moment, however, we need to use one half the difference of the squares of the upper and lower functions:

$$M_y = \int_0^3 \frac{1}{2}(3x)^2 - \frac{1}{2}\left(x^2\right)^2 dx = \frac{1}{2}\int_0^3 9x^2 - x^4 dx = \frac{1}{2}\left[3x^3 - \frac{x^5}{5}\right]_0^3 = \frac{1}{2}\left(81 - \frac{243}{5}\right) - (0-0) = \frac{81}{5}$$

Thus, the center of mass of the above region is $\left(\frac{M_x}{A}, \frac{M_y}{A}\right) = \left(\frac{3}{2}, \frac{18}{5}\right)$.

2. **(15 points)** *Evaluate the following integrals, or if they cannot be evaluated, demonstrate why not.*

   (a) **(7 points)** $\int_{-2}^4 \frac{1}{x} dx$

   The function $\frac{1}{x}$ has a discontinuity at $x = 0$, so the above integral is improper and must be rephrased as a sum of limites of definite integrals:

$$\begin{aligned} \int_{-2}^4 \frac{1}{x} dx &= \lim_{b\to 0^-} \int_{-2}^b \frac{1}{x} dx + \lim_{a\to 0^+} \int_a^4 \frac{1}{x} dx \\ &= \lim_{b\to 0^-} \ln|x|\big]_{-2}^b + \lim_{a\to 0^+} \ln|x|\big]_a^4 \\ &= \lim_{b\to 0^-} \ln|b| - \ln 2 + \lim_{a\to 0^+} \ln 4 - \ln|a| \\ &= \ln 4 - \ln 2 + \lim_{b\to 0^-} \ln|b| - \lim_{a\to 0^+} \ln|a| \end{aligned}$$

   Since zero has no natural logarithm, or and there is not even a limit to the natural logarithm as its argument approaches zero, neither of the limits shown above exist, so the integral is divergent.

(b) **(8 points)** $\int_5^\infty \frac{1}{\sqrt[3]{x-4}}dx$
Using limits to rephrase this improper integral:

$$\begin{aligned}
\int_5^\infty \frac{1}{\sqrt[3]{x-4}}dx &= \lim_{b\to\infty}\int_5^b (x-4)^{-1/3}dx \\
&= \lim_{b\to\infty} \frac{(x-4)^{2/3}}{2/3}\Big]_5^b \\
&= \lim_{b\to\infty} \frac{(b-4)^{2/3}}{2/3} - \frac{(5-4)^{2/3}}{2/3}
\end{aligned}$$

However, as b grows without bound, so does $(b-4)^{2/3}$, so this integral is divergent.

3. **(15 points)** *Consider the function* $f(x) = \begin{cases} 0 \text{ for } x < 4 \\ \frac{k}{x^{5/2}} \text{ for } x \geq 4 \end{cases}$ *with k a constant.*

(a) **(6 points)** *Find a value of k such that $f(x)$ is a probability distribution function.*
A cursory inspection reveals that this function is non-negative throughout if k is positive: 0 is non-negative everywhere, and $\frac{1}{x^{5/2}}$ is non-negative in its entire domain. The critical property to demonstrate that this function is a probability distribution function is simply that $\int_{-\infty}^{\infty} f(x)dx = 1$. We can simplify this somewhat by ignoring the region on which $f(x)$ is zero, so that $\int_{-\infty}^{\infty} f(x) = \int_4^\infty f(x)dx$. We evaluate this as such:

$$\begin{aligned}
\int_4^\infty f(x)dx &= \int_4^\infty \frac{k}{x^{5/2}}dx \\
&= \lim_{b\to\infty}\int_4^b \frac{k}{x^{5/2}}dx \\
&= \lim_{b\to\infty} \frac{-2k}{3x^{3/2}}\Big]_4^b \\
&= \lim_{b\to\infty} \frac{2k}{3b^{3/2}} + \frac{2k}{3\cdot 4^{3/2}} \\
&= 0 + \frac{2k}{24}
\end{aligned}$$

so for $\frac{2k}{24}$ to equal 1, we would choose $k = 12$.

(b) **(6 points)** *For a random variable X described by the above probability distribution function, find the average value of X.*
The expected value (or average value) of a probability distribution function $f(x)$ is $\int_{-\infty}^{\infty} xf(x)dx$. We may ignore locations where the integrand is zero, so this can be

simplified to $\int_4^\infty xf(x)dx$:

$$\begin{aligned}\int_4^\infty xf(x)dx &= \int_4^\infty x\frac{12}{x^{5/2}}dx \\ &= \lim_{b\to\infty}\int_4^b \frac{12}{x^{3/2}}dx \\ &= \lim_{b\to\infty}\left.\frac{-24}{x^{1/2}}\right]_4^b \\ &= \lim_{b\to\infty}\frac{-24}{b^{1/2}}+\frac{24}{4^{1/2}} \\ &= 0+\frac{24}{2}=12\end{aligned}$$

(c) **(3 points)** *For a random variable X described by the above probability distribution function, find $P(X \le 9)$.*

This probability will be simply $\int_4^9 f(x)dx$. It is technically $\int_{-\infty}^9 f(x)dx$, but we can ignore the region over which the function is zero.

$$\int_4^9 f(x)dx = \int_4^9 \frac{12}{x^{5/2}}dx = \left.\frac{-8}{x^{3/2}}\right]_4^9 = \frac{-8}{9^{3/2}}+\frac{8}{4^{3/2}} = \frac{-8}{27}+1 = \frac{19}{27}$$

4. **(10 points)** *Consider the curve $y = e^x + 4$ between the points $(0, 4)$ and $(2, 4 + e^2)$.*

(a) **(4 points)** *Construct, but do not evaluate, an integral representing the length of this curve.*

The arclength expression $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ evaluates in this case to $\sqrt{1+(e^x)^2}$, so the arclength is

$$\int_0^4 \sqrt{1+e^{2x}}dx$$

(b) **(3 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the vertical line $x = -3$.*

Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1+e^{2x}}dx$, being spun around circles of radius $x+3$, since the horizontal distance between the line $x = -3$ and the point $(x, e^x + 4)$ is $x + 3$; thus the differential area traced out is $2\pi(x + 3)\sqrt{1+e^{2x}}dx$, so the integral to compute the total surface area is

$$\int_0^4 2\pi(x+3)\sqrt{1+e^{2x}}dx$$

(c) **(3 points)** *Construct, but do not evaluate, an integral representing the surface area of the surface produced by rotating this curve around the x-axis.*

Such a revolution would involve arcs given, as shown in part (a), to be of differential length $\sqrt{1+e^{2x}}dx$, being spun around circles of radius $e^x + 4$, since the vertical distance between the line $y = 0$ and the point $(x, e^x + 4)$ is simply $e^x + 4$; thus the differential area traced out is $2\pi(e^x + 4)\sqrt{1 + e^{2x}}dx$, so the integral to compute the total surface area is

$$\int_0^4 2\pi(e^x + 4)\sqrt{1 + e^{2x}}dx$$

5. **(15 points)** *Perform the approximations shown below.*

   (a) **(5 points)** *Using Simpson's rule with $n = 6$, approximate $\int_1^4 \frac{1}{x}dx$. You need not arithmetically simplify your result.*

   Since the interval has length 3 and $n = 6$, the choice of Δx is $\frac{3}{6} = \frac{1}{2}$. Thus, we will need to sample the integrand at the seven points $x = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$. Assembling these into Simpson's rule, we would find that the approximation is:

$$\frac{1/2}{3}\left(\frac{1}{1} + \frac{4}{3/2} + \frac{2}{2} + \frac{4}{5/2} + \frac{2}{3} + \frac{4}{7/2} + \frac{1}{4}\right)$$

   If this were simplified, we would have $\frac{3497}{2520}$, which is not a particularly bad rational estimate for the actual integral, which is $\ln 4$.

   (b) **(5 points)** *Using the trapezoidal rule with $n = 6$, approximate $\int_1^4 \frac{1}{x}dx$. You need not arithmetically simplify your result.*

   Since the interval has length 3 and $n = 6$, the choice of Δx is $\frac{3}{6} = \frac{1}{2}$. Thus, we will need to sample the integrand at the seven points $x = 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4$. Assembling these into the midpoint rule, we would find that the approximation is:

$$\frac{1/2}{2}\left(\frac{1}{1} + \frac{2}{3/2} + \frac{2}{2} + \frac{2}{5/2} + \frac{2}{3} + \frac{2}{7/2} + \frac{1}{4}\right)$$

   If this were simplified, we would have $\frac{787}{560}$, which is not a particularly bad rational estimate for the actual integral, which is $\ln 4$.

   (c) **(5 points)** *Using the midpoint rule with $n = 6$, approximate $\int_1^4 \frac{1}{x}dx$. You need not arithmetically simplify your result.*

   Since the interval has length 3 and $n = 6$, the choice of Δx is $\frac{3}{6} = \frac{1}{2}$. Thus, we will need to sample the integrand at the six interval midpoints $x = 1.25, 1.75, 2.25, 2.75, 3.25, 3.75$. Assembling these into the midpoint rule, we would find that the approximation is:

$$\frac{1}{2}\left(\frac{1}{1.25} + \frac{1}{1.75} + \frac{1}{2.25} + \frac{1}{2.75} + \frac{1}{3.25} + \frac{1}{3.75}\right)$$

   If this were simplified, we would have $\frac{62024}{45045}$, which is not a particularly bad rational estimate for the actual integral, which is $\ln 4$.

6. **(15 points)** *Consider the curve given by the parametric equations $x = t^2$ and $y = t - t^2$.*

   (a) **(8 points)** *Find the area under the curve between $t = 0$ and $t = 1$. Your answer need not be arithmetically simplified.*

   Using the parametric formula for area:

$$\int_0^1 y\frac{dx}{dt}dt = \int_0^1 (t - t^2)(2t)dt = \int_0^1 2t^2 - 2t^3 dt = \frac{2}{3}t^3 - \frac{1}{2}t^4\Big]_0^1 = \left(\frac{2}{3} - \frac{1}{2}\right) - (0 - 0) = \frac{1}{6}$$

(b) **(7 points)** *Construct, but do not evaluate, an integral representing the arclength of the curve between $t = 0$ and $t = 1$.*

Using the parametric formula for arclength:

$$\int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^1 \sqrt{(2t)^2 + (1-2t)^2} dt = \int_0^1 \sqrt{8t^2 - 4t + 1} dt$$