1. **(25 points)** *Identify each of the following series as conditionally convergent, absolutely convergent, or divergent:*

   (a) **(7 points)** $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.
   The presence of a factorial strongly suggests the ratio test, and if in fact we inspect the ratio of successive terms:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{(n+1)!}}{\frac{n^2}{n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^2 n!}{n^2(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^2}{n^2(n+1)} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

   Since the result of the ratio test is less than 1, this series is *absolutely convergent.*

   (b) **(6 points)** $\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n$.
   Since each term is an nth power, the root test is indicated in this situation. Looking at the limit of the nth root of the nth term of the series:

$$\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{1}{n+1}\right)^n \right|} = \lim_{n\to\infty} \frac{1}{|n+1|} = 0$$

   Since the result of the root test is less than 1, this series is *absolutely convergent.*

   (c) **(6 points)** $\sum_{n=1}^{\infty} \frac{3^n}{n}$.
   There are several approaches which can be taken here, but the ratio test is perhaps the most straightforward:

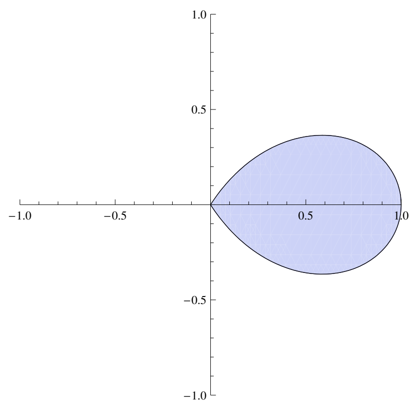
$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{n+1}}{\frac{3^n}{n}} \right| = \lim_{n\to\infty} \frac{3^{n+1} n}{3^n(n+1)} = \lim_{n\to\infty} \frac{3n}{(n+1)} = \lim_{n\to\infty} \frac{3n}{n} = 3$$

   Since the result of the ratio test is greater than 1, this series is *divergent.*

   (d) **(6 points)** $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$.
   This is an alternating p-series with $p = 4$, which is known to converge (alternatively: it is an alternating series whose terms' absolute values decreasing and tend towards zero, and thus the series converges by the alternating series test). In addition, taking the absolute value of each term yields the positive p-series with $p = 4$, which is known to converge (by either familiarity with p-series, or by the integral test). Since this series' termwise absolute value converges, the series is *absolutely convergent.*

2. **(15 points)** *Consider the curve given by the polar equation $r = 1 - \theta^2$ between $\theta = -1$ and $\theta = 1$ (shown below):*

   (a) **(9 points)** *What is the area of the shaded region? You need not arithmetically simplify your answer.*



By the polar area formula, the shaded region's area is given by the integral:

$$\int_{-1}^{1} \frac{1}{2}(1-\theta^2)^2 d\theta = \int_{-1}^{1} \frac{1-2\theta^2+\theta^4}{2} = \frac{\theta}{2} - \frac{\theta^3}{3} + \frac{\theta^5}{10}\Big]_{-1}^{1} = \left(\frac{1}{2}-\frac{1}{3}+\frac{1}{10}\right)-\left(\frac{-1}{2}+\frac{1}{3}-\frac{1}{10}\right) = \frac{8}{15}$$

(b) **(6 points)** *Find the arclength of this curve.*

By the polar arclength formula, the length of the curve is given by the integral:

$$\begin{aligned}\int_{-1}^{1}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta &= \int_{-1}^{1}\sqrt{(1-\theta^2)^2+(-2\theta)^2}d\theta \\ &= \int_{-1}^{1}\sqrt{\theta^4+2\theta^2+1}d\theta \\ &= \int_{-1}^{1}\theta^2+1d\theta \\ &= \frac{\theta^3}{3}+\theta\Big]_{-1}^{1} = \left(\frac{1}{3}+1\right)-\left(\frac{-1}{3}-1\right) = \frac{8}{3}\end{aligned}$$

3. **(15 points)** *Determine whether each of the following series converges, and justify your claim:*

(a) **(5 points)** $\sum_{n=1}^{\infty} \frac{1+e^{-n}}{\sqrt{n}}$.

Since $1+e^{-n} > 1$, $\frac{1+e^{n}}{\sqrt{n}} > \frac{1}{\sqrt{n}} > 0$; since $\sum_{n=0}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p=\frac{1}{2}$, it diverges. Thus, by the comparison test, the series with larger terms $\sum_{n=1}^{\infty}\frac{1+e^{-n}}{\sqrt{n}}$ also diverges.

Limit comparison would also suffice to show equivalent convergence-status of these two series.

(b) **(5 points)** $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$.

Over the long term, $\frac{n}{n^2+1}$ resembles $\frac{n}{n^2}=\frac{1}{n}$. We make this resemblance explicit via limit comparison:

$$\lim_{n\to\infty}\frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1$$

Since limit comparison yields a nonzero real number, $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ must have the same convergence status. The latter is the well-known harmonic series, which diverges, so the former series also diverges.

Divergence may also be shown directly via the integral test.

(c) **(5 points)** $\frac{1}{1} - \frac{1}{8} + \frac{1}{27} - \frac{1}{64} + \frac{1}{125} - \frac{1}{216} + \cdots$.

This series may be concisely written as $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$, which can be seen to converge by either the alternating series test or by virtue of being an alternating p-series with $p = 3$; alternating p-series converge from any positive p.

4. **(15 points)** *Answer the following questions about polar coordinates.*

(a) **(7 points)** *Express the Cartesian formula $y = x^3$ in terms of polar coordinates, with r expressed as a function of θ.*

Using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$, simple algebra serves to express r as a function of θ:

$$\begin{aligned} y &= x^3 \\ r\sin\theta &= (r\cos\theta)^3 \\ r\sin\theta &= r^3\cos^3\theta \\ \frac{\sin\theta}{\cos^3\theta} &= r^2 \\ \sqrt{\frac{\sin\theta}{\cos^3\theta}} &= r \end{aligned}$$

(b) **(8 points)** *Express the polar formula $r = \frac{3\sin\theta}{\cos^2\theta}$ in terms of Cartesian coordinates, with y expressed as a function of x.*

Direct substitutions for θwill produce extremely unpleasant results, but the more palatable substitutions $r\sin\theta = y$ and $r\cos\theta = x$ can be invoked after a modicum of algebraic manipulation:

$$\begin{aligned} r &= \frac{3\sin\theta}{\cos^2\theta} \\ r\cos^2\theta &= 3\sin\theta \\ r^2\cos^2\theta &= 3r\sin\theta \\ x^2 &= 3y \\ \frac{x^2}{3} &= y \end{aligned}$$

5. **(15 points)** *Answer the following two questions about the power series $\sum_{n=0}^{\infty} \frac{2n(x-2)^n}{5^n}$.*

(a) **(8 points)** *Find its interval of convergence.*

We start by applying the ratio test to find the interior of the interval:

$$\lim_{n\to\infty} \left| \frac{\frac{2(n+1)(x-2)^{n+1}}{5^{n+1}}}{\frac{2n(x-2)^n}{5^n}} \right| = \lim_{n\to\infty} \left| \frac{2(n+1)(x-2)^{n+1}5^n}{5^{n+1}2n(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-2)}{5n} \right| = \left| \frac{x-2}{5} \right|$$

The criterion for convergence is thus that $\left|\frac{x-2}{5}\right| \lesssim 1$ (using nonstandard notation to denote the possibility but not the certainty that this series converges when the ratio test yields a value of 1). An absolute value is less than 1 if the expression in question is between -1 and 1, thus the interval of convergence is

$$\begin{aligned}-1 &\lesssim \frac{x-2}{5} \lesssim 1\\ -5 &\lesssim x-2 \lesssim 5\\ -3 &\lesssim x \lesssim 7\end{aligned}$$

So our interval of convergence ranges from -3 to 7, possibly including the endpoints, which must be tested separately.

When $x = -3$, the series evaluates to $\sum_{n=0}^{\infty} \frac{2n(-5)^n}{5^n} = \sum_{n=0}^{\infty}(-1)^n 2n$, which can easily be seen to be an alternating series. However, as the terms do not tend towards zero, it is clearly a divergent series.

When $x = 7$, the series evaluates to $\sum_{n=0}^{\infty} \frac{2n \cdot 5^n}{5^n} = \sum_{n=0}^{\infty} 2n$, which can easily be seen to be a positive series with increasing terms. This will clearly not converge, as the terms do not tend towards zero.

Since the series does not converge at the endpoints of the interval of convergence, we now know the interval to be specifically $-3 < x < 7$, which can also be written in interval notation as $(-3, 7)$.

(b) **(7 points)** *Find a function which is equal to it.*

We might note produce this series using the following sequences of transformations of the ordinary geometric series:

$$\begin{aligned}\sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} & &\\ \sum_{n=0}^{\infty}(n+1)x^n &= \frac{1}{(1-x)^2} & &\left(\frac{d}{dx}\right)\\ \sum_{n=0}^{\infty} 2nx^n &= \frac{2x}{(1-x)^2} & &(\cdot 2x)\\ \sum_{n=0}^{\infty} 2n\left(\frac{x-2}{5}\right)^n &= \frac{2\frac{x-2}{5}}{(1-\frac{x-2}{5})^2} & &\left(x \mapsto \frac{x-2}{5}\right)\\ \sum_{n=0}^{\infty} \frac{2n(x-2)^n}{5^n} &= \frac{10x-20}{(7-x)^2} & &\end{aligned}$$

The last line is cleanup and is optional.

6. **(15 points)** *Does each of the following converge? If so, what does it converge to and why? If not, why not?*

(a) **(5 points)** *The sequence* $\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{5}}, \ldots\}$.

This sequence has general term $a_n = \frac{1}{\sqrt{n}}$. By analogy to the function $f(x) = \frac{1}{\sqrt{x}}$, one can see that as $n \to \infty$, $a_n \to 0$, so this sequence converges to 0.

(b) **(5 points)** *The sequence* $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots\}$.

This sequence has general term $a_n = \frac{n-1}{n}$; since $\lim_{x\to\infty} \frac{x-1}{x} = 1$, the sequence likewise converges to 1.

(c) **(5 points)** *The series* $1 + \frac{4}{3} + \frac{16}{9} + \frac{64}{27} + \frac{256}{81} + \ldots$.

This series can be easily determined by inspection to be geometric with common ratio $\frac{4}{3}$. Since $|\frac{4}{3}| > 1$, this series diverges.