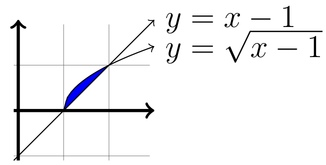
1. **(5 points)** *Find the area of the region between the curves $y = \sqrt{x-1}$ and $y = x - 1$. You may find it helpful to sketch the curves.*



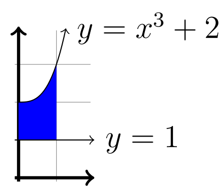
As we can see, the curve $y = \sqrt{x-1}$ lies over the curve $y = x - 1$ between the x-values of 1 and 2; thus, the integral we wish to set up is

$$\int_1^2 \sqrt{x-1} - (x-1)dx$$

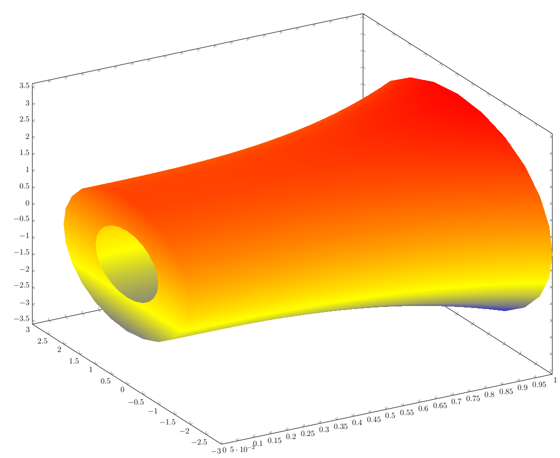
which can be solved easily with the (implicit) substitution $u = x - 1$:

$$\int_1^2 \sqrt{x-1} - (x-1)dx = \frac{(x-1)^{3/2}}{3/2} - \frac{(x-1)^2}{2}\Bigg]_1^2 = \left(\frac{2}{3} - \frac{1}{2}\right) - (0-0) = \frac{1}{6}$$

2. **(12 points)** *Below, we are considering the region enclosed by the curves $y = x^3 + 2$, $y = 1$, $x = 0$, and $x = 1$. You may find it useful to sketch the region.*



(a) **(6 points)** *Find the volume of the solid obtained by rotating this region around the x-axis.*

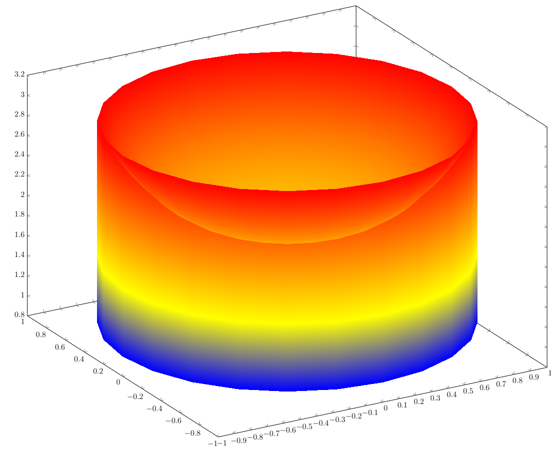


Above, for purposes of illustration, the solid is drawn. It can be thought of as a series of washers (or rings, or annuli) between $x = 0$ and $x = 1$ with outer radius $x^3 + 2$ and inner radius 1, so the integral to be performed is:

$$\int_0^1 \pi\big((x^3+2)^2 - 1^2\big)dx = \int_0^1 \pi(x^6 + 4x^3 + 3)dx = \pi\left[\frac{x^7}{7} + x^4 + 3x\right]_0^1 = \frac{29\pi}{7}$$

It is possible, but inadvisable, to use the sum of two solids decomposable into cylindrical shells to get this answer; the formulation to do so would be $\int_1^2 2\pi y(1-0)dy + \int_2^3 2\pi y(1 - \sqrt[3]{y-2})dy$.

(b) **(6 points)** *Find the volume of the solid obtained by rotating this region around the y-axis.*



Above, for purposes of illustration, the solid is drawn. It can be thought of as a series of cylindrical shells whose radii range from $x = 0$ to $x = 1$ and whose heights are $(x^3 + 2) - 1$, so the integral to be performed is:

$$\int_0^1 2\pi x(x^3 + 1)dx = \int_0^1 \pi(2x^4 + 2x)dx = \pi\left[\frac{2x^5}{5} + x^2\right]_0^1 = \frac{7\pi}{5}$$

It is possible, but inadvisable, to use the sum of two solids decomposable into washers to get this answer; the formulation to do so would be $\int_1^2 \pi(1^2 - 0^2)dy + \int_2^3 \pi y(1^2 - (y-2)^{3/2})dy$.

3. **(3 points)** *Determine the average value of* $f(x) = \frac{1}{x^2}$ *on the interval* $[1, 3]$.

By the averaging formula, the average value of $f(x)$ on this interval is

$$\frac{\int_1^3 \frac{1}{x^2}dx}{3-1} = \frac{\left.\frac{-1}{x}\right]_1^3}{2} = \frac{\frac{-1}{3} - \frac{-1}{1}}{2} = \frac{1}{3}$$

4. **(2 point bonus)** *A solid has a height of h, and top and bottom faces which are the same shape but different sizes, having respective areas a and A. These faces are connected by straight lines. On the back of this sheet, determine the volume of the solid in terms of h, a, and A.*

The shape described is basically a truncated pyramid (or, if the top and bottom faces are circles, a *frustum*). We know that linear distances among these faces scale linearly from bottom to top, so that areas scale proportionally to the square of some linear function. Thus, we know that the cross-sectional area $\mathcal{A}$ at a height z is given by some formula of the form $(mz + b)^2$. Since $b^2 = \mathcal{A}(0) = A$ and $(mh + b)^2 = \mathcal{A}(h) = a$, we know that $b = \sqrt{A}$ and $m = \frac{\sqrt{a} - \sqrt{A}}{h}$. Thus, we may calculate our volume to be:

$$\int_0^h \left(\frac{\sqrt{a} - \sqrt{A}}{h}z + \sqrt{A}\right)^2 dz$$

and using the (implicit or explicit) substitution $u = \frac{\sqrt{a} - \sqrt{A}}{h}z + \sqrt{A}$, we may calculate this integral to be

$$\left.\frac{h}{\sqrt{a} - \sqrt{A}} \cdot \frac{1}{3}\left(\frac{\sqrt{a} - \sqrt{A}}{h}z + \sqrt{A}\right)^3\right]_0^h = \frac{h}{3(\sqrt{a} - \sqrt{A})}(a^{3/2} - A^{3/2}) = \frac{h(a + \sqrt{aA} + A)}{3}$$