1. **(10 points)** *Let us call a number "good" if all of its digits are the same parity (i.e. all odd or all even). So, for instance, 62880 and 31713 are both good, but 61407 is not, since 6, 4, and 0 have different parities from 1 and 7.*

   (a) **(5 points)** *Determine the number of 3-digit "good" numbers; note that numbers, when conventionally written, do not have a leading 0; e.g. "024" is not a three-digit number.*

   A "good" number might be either even or odd. If it is even, there are four choices for the first digit (2, 4, 6, or 8) and five for the second and third digits (0, 2, 4, 6, 8), so there are $4 \cdot 5 \cdot 5 = 100$ even good three-digit numbers.

   Likewise, an odd good number might have any of five different possible choices for each of the three digits, so there are $5 \cdot 5 \cdot 5 = 125$ odd good three-digit numbers.

   Putting these two sets together, we see that there are $100 + 125 = 225$ good three-digit numbers in all.

   (b) **(5 points)** *Find a formula in terms of $n$ for the number of $n$-digit "good" numbers.*

   The same logic can be applied as above, only instead of having three digits, there are now $n$ digits; thus we can compute that there are $4 \cdot 5^{n-1}$ even good numbers and $5^n$ odd good numbers for a total of $4 \cdot 5^{n-1} + 5^n = 9 \cdot 5^{n-1}$ good $n$-digit numbers.

2. **(10 points)** *A modified deck of cards contains five suits and ten cards (numbered 1–10) in each suit. A "straight" is defined as in poker as a hand in which the five cards are numerically consecutive, and of any suit (ignore, for the purposes of this problem, the poker convention that a hand of this sort could also be a "straight flush" or "royal flush" instead). How many different five-card hands (which are not ordered) are straights?*

   The numbers on the cards must be one of six choices: 1–5, 2–6, 3–7, 4–8, 5–9, or 6–10. After we have chosen the numbers, we choose a suit for each card; there are five choices for each card, so in total we have $6 \cdot 5^5 = 18750$ different possible straights.

3. **(10 points)** *Let us consider license plates which consist of two consecutive groups of either three letters (from a pool of 26 possible) or three numbers (from a pool of 10 possible); e.g. 134-WRG, EEE-EEK, and 852-584 are all possible license plates. However, to avoid confusion, you forbid either group of 3 to consist entirely of the numbers "0", "1", and "5", or the letters "O", "I", and "S" (so we don't allow 115-RTG, because it might be misread as IIS-RTG). How many license plates are possible according to this scheme (you may leave the answer as an unsimplified arithmetic expression, if you like).*

   Looking at a single group of three, there are $26^3$ ways to build it with letters, and $10^3$ ways to build it with numbers. Of those $(26^3 + 10^3)$ possibilities, we must reject $3^3$ letter-sequences made with only O, I, and S, and $3^3$ number-sequences made with only 0, 1, and 5. Thus a single group of 3 can be made in any of $(26^3 + 10^3 - 2 \cdot 3^3)$ ways. Since we build a license plate by selecting two such groups, there are $(26^3 + 10^3 - 2 \cdot 3^3)^2 = 345,997,201$ different license plates possible.

4. **(10 points)** *Suppose we build a random four-letter "word" by placing in order a random selection from the 21 consonants, a random selection from the 5 vowels, another random consonant, and then another random vowel (most words we produce this way will be nonsense words like "LEFE"). What is the probability that we make a word in which all four letters are different, like "XOFA"?*

   A probability when choosing randomly is simply the ratio between the number of desirable results and the size of the entire result pool. We can build an arbitrary "word" in any of $21 \cdot 5 \cdot 21 \cdot 5 = 11025$ ways, and all of our letters can be distinct in $21 \cdot 5 \cdot 20 \cdot 4 = 8400$ ways, since for the second consonant and vowel we must choose a different letter from the one already chosen. Thus the probability is $\frac{11025}{8400} = \frac{16}{21} \approx 76\%$.

> Musica est exercitium arithmeticae occultum nescientis se numerare animi. [Music is an arithmetical exercise of the soul, which is unaware that it is counting.]
>
> —Gottfried Leibniz