1. **(10 points)** *Prove that for every positive integer n,*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{\ell=0}^{n} \sum_{r=0}^{\lfloor \frac{n-\ell}{2} \rfloor} \binom{n}{\ell}\binom{n-\ell}{r}\binom{n-\ell-r}{r}.$$

For example:

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20 =$$
$$\binom{3}{0}\binom{3}{0}\binom{3}{0} + \binom{3}{0}\binom{3}{1}\binom{2}{1} + \binom{3}{1}\binom{2}{0}\binom{2}{0} + \binom{3}{1}\binom{2}{1}\binom{1}{1} + \binom{3}{2}\binom{1}{0}\binom{1}{0} + \binom{3}{3}\binom{0}{0}\binom{0}{0}$$

(Hint: you probably want to prove this combinatorially: the left side can be thought of as enumerating pairs of subsets of $\{1, 2, \ldots, n\}$ with a particular property. Can you characterize each pair in other ways, and count those characterizations?)

For each value k, $\binom{n}{k}^2$ counts the pairs (A, B) of subsets of $\{1, 2, 3, \ldots, n\}$ such that $|A| = |B| = k$. Since we allow k to range over all possible values, the left side will thus enumerate the pairs of subsets of $\{1, 2, 3, \ldots, n\}$ where $|A| = |B|$.

On the other hand, the expression on the right side is $\binom{n}{\ell}\binom{n-\ell}{r}\binom{n-\ell-r}{r}$, which we might alternatively write as the multinomial $\binom{n}{\ell,r,r,n-\ell-2r}$; it can be thought of as the selection of a triple of three *disjoint* subsets (R, S, T) of $\{1, \ldots, n\}$ with $|R| = \ell$, $|S| = r$, and $|T| = r$; the summation simply guarantees that R, S, and T take on all possible values such that $|R| + |S| + |T| \leq n$, so all we are really selecting on the right is three disjoint sets R, S, and T such that $|S| = |T|$. We now wish to show that this structure is fundamentally the same as that on the left side. We may do this via a bijection: let $A = R \cup S$ and $B = R \cup T$, or, conversely, let $R = A \cap B$, $S = A - B$, and $T = B - A$.

In summary, we show these two values are identical by showing that they enumerate two distinct processes for building equal-size subsets of $\{1, 2, 3, \ldots, n\}$: we could select a size for A and B and build two subsets of that size, or we could select a size for the overlapping and non-overlapping portions, and select the overlap, then A's uniquely-held elements, and then B's uniquely-held elements.

2. **(10 points)** *Prove combinatorially that for positive integers m, w, and k, $\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}$. (Hint: determine what the right side counts, and then divide those objects into $k+1$ classes whose sizes are on the left, based on their properties.*

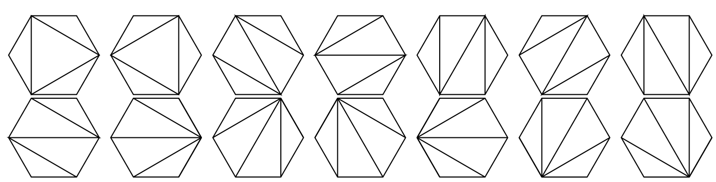
The right side of this expression clearly counts the possible k-element subsets of $\{1, 2, 3, \ldots, m + w\}$. To acquire the left side, let us consider how such a k-element subset S might have its elements divided between the smaller numbers $\{1, 2, 3, \ldots, m\}$ and the larger numbers $\{m + 1, m + 2, \ldots, m + w\}$. Plausibly anywhere between 0 and k of the elements might belong to the former set; let's call the number of such elements j. Then for any specific j, we could build S by selecting j elements from $\{1, 2, 3, \ldots, m\}$ in any of $\binom{m}{j}$ ways, and selecting the remaining $k - j$ elements from $\{m+1, m+2, \ldots, m+w\}$ in any of $\binom{w}{k-j}$ ways. Thus for a fixed j we might build the set S in any of $\binom{m}{j}\binom{w}{k-j}$ ways. However, since any value of j is possible, we must add up this calculation for each j from 0 to k, to get the desired count of $\sum_{j=0}^{k}\binom{m}{j}\binom{w}{k-j}$.

3. **(10 points)** *Suppose that a teacher wishes to distribute 25 identical pencils to Ahmed, Barbara, Casper, and Dieter such that Ahmed and Dieter receive at least one pencil each, Casper receives no more than five pencils, and Barbara receives at least four pencils. In how many ways can such a distribution be made?*

   We might pre-emptively assign one pencil to each of Ahmed and Dieter and four to Barbara; we now have simply the task of freely assigning the remaining 19 pencils to the four students giving no more than five to Caspar. Ignoring this last restriction for the time being, the balls-and-walls paradigm allows us to consider this as the placement of three partitions among 22 positions which can be achieved in $\binom{22}{3}$ ways. However, we now want to *exclude* from consideration any distribution which gives 6 or more pencils to Casper; to count these, we can pre-emptively give 6 pencils to Casper and count the ways to assign the remaining 13 pencils, and again our balls-and-walls paradigm provides a count of $\binom{16}{3}$. Since this most recent calculation is the number of placements to exclude, our final count will be $\binom{22}{3}-\binom{16}{3}=980$.

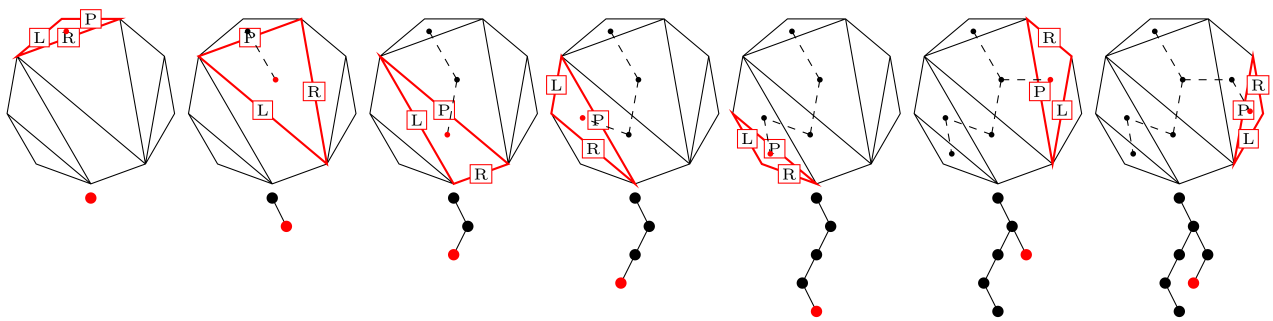
4. **(10 points)** *The Catalan number C_n enumerates, among other things, the number of ways to nest n pairs of parentheses, the number of ways to build a binary tree with n nodes, the number of Dyck paths from $(0,0)$ to $(0,2n)$, and the number of ways to build a binary tree where each node with children has 2 children, and which has $n+1$ leaves.*

   Show with an explicit bijection *to one of the structures described above, that the triangulations by noncrossing diagonals of a convex $(n+2)$-gon are also enumerated by C_n. For instance, $C_4 = 14$, and below are the 14 triangulations of a hexagon:*

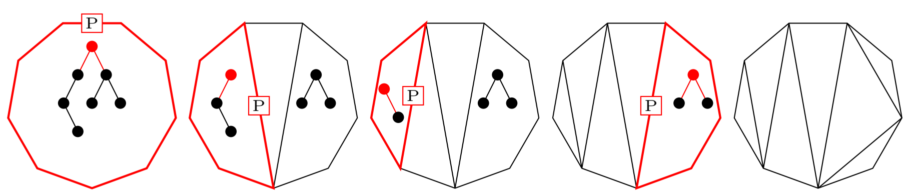
   

   Several explicit bijections are possible; here I show one of the more straightforward ones. For purposes of illustration the techniques below will be illustrated with a nonagon and a 7-node binary tree.

   Let us choose a nonagon to triangulate in which one edge lies at the top; we will call this the "root edge" and the triangle containing it as the "root triangle", which will eventually resolve into our root node. The root triangle has two other edges, one clockwise from the root edge, and one counterclockwise from the root edge; we shall call these the "right" and "left" edges respectively. If either edge is on the outside of the nonagon we consider it to represent a non-child; either which is interior to the nonagon will become the "parent edge" of whatever other triangle it borders. Now each "child triangle" has a "parent edge" and two other edges. Just as with the root triangle, we label these two edges with the clockwise one being called "right" and the counterclockwise one "left". Repeating this until all triangles are labeled, we see that each triangle (except the root) has a parent, and that the triangle is the left or right child of that parent. This means we can associate each triangle in the triangulation with a single node of a binary tree. Below I show how this might work for a specific triangulation, labeling the currently-being-processed triangle in bold red, and labeling the parent edge, left edge and right edge thereof with "P", "L", and "R" respectively. For consistency of notation we will call the root edge the "parent edge" of the root triangle.



Conversely, starting with a tree with ℓ children on the left and r children on the right, we can iteratively build triangles, starting from our parent edge, which have an $(\ell+2)$-gon on the left and an $(r+2)$-gon on the right *relative to our parent edge*, as such:



שתי אבנים בונות שני בתים: שלש אבנים בונות ששה בתים: ארבע אבנים בונות ארבעה ועשרים בתים: חמש אבנים בונות מאה ועשרים בתים: שש אבנים בונות שבע מאות ועשרים בתים: שבע אבנים בונות חמשת אלפים וארבעים בתים: מכאן ואילך צא וחשב מה שאין הפה יכול לדבר ואין האזן יכולה לשמע

— ספר יצירה פרק ד משנה טז

[Two stones (or letters) build two houses (or words), three stones build six houses, four stones build twenty-four houses, five stones build one hundred twenty houses, six stones build seven and twenty houses, seven stones build five thousand forty houses; thenceforth are numbers which the mouth can not speak and the ear can not hear.]

—Sefer Yetzira, Chapter 4, Verse 16