1. **(10 points)** Prove that for every positive integer n,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{\ell=0}^{n} \sum_{r=0}^{\lfloor \frac{n-\ell}{2} \rfloor} \binom{n}{\ell} \binom{n-\ell}{r} \binom{n-\ell-r}{r}.$$

For example:

$$\binom{3}{0}^2 + \binom{3}{1}^2 + \binom{3}{2}^2 + \binom{3}{3}^2 = 20 =$$

$$\binom{3}{0}\binom{3}{0}\binom{3}{0}+\binom{3}{0}\binom{3}{1}\binom{2}{1}+\binom{3}{1}\binom{2}{0}\binom{2}{0}+\binom{3}{1}\binom{2}{1}\binom{1}{1}+\binom{3}{2}\binom{1}{0}\binom{1}{0}+\binom{3}{3}\binom{0}{0}\binom{0}{0}$$

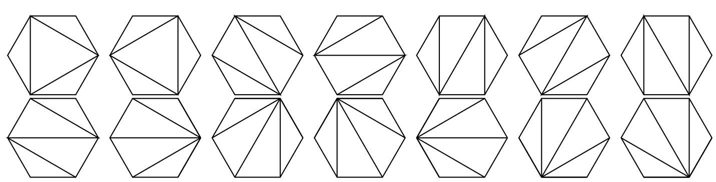
(Hint: you probably want to prove this combinatorially: the left side can be thought of as enumerating pairs of subsets of $\{1, 2, \ldots, n\}$ with a particular property. Can you characterize each pair in other ways, and count those characterizations?)

2. **(10 points)** Prove combinatorially that for positive integers m, w, and k, $\sum_{j=0}^{k} \binom{m}{j}\binom{w}{k-j} = \binom{m+w}{k}$. (Hint: determine what the right side counts, and then divide those objects into $k+1$ classes whose sizes are on the left, based on their properties.

3. **(10 points)** Suppose that a teacher wishes to distribute 25 identical pencils to Ahmed, Barbara, Casper, and Dieter such that Ahmed and Dieter receive at least one pencil each, Casper receives no more than five pencils, and Barbara receives at least four pencils. In how many ways can such a distribution be made?

4. **(10 points)** The Catalan number C_n enumerates, among other things, the number of ways to nest n pairs of parentheses, the number of ways to build a binary tree with n nodes, the number of Dyck paths from $(0, 0)$ to $(0, 2n)$, and the number of ways to build a binary tree where each node with children has 2 children, and which has $n + 1$ leaves.

Show with an *explicit bijection* to one of the structures described above, that the triangulations by noncrossing diagonals of a convex $(n + 2)$-gon are also enumerated by C_n. For instance, $C_4 = 14$, and below are the 14 triangulations of a hexagon:



שתי אבנים בונות שני בתים: שלש אבנים בונות ששה בתים: ארבע אבנים ב ונות ארבעה ועשרים בתים: חמש אבנים בונות מאה ועשרים בתים: שש אבנים בונות ש בע מאות ועשרים בתים: שבע אבנים בונות חמשת אלפים וארבעים בתים: מכאן ו אילך צא וחשוב מה שאין הפה יכול לדבר ואין האוזן יכולה לשמוע

—ספר היצירה פרק ד׳ משנה ט״ז

 due September 21, 2015