1. **(10 points)** *Using induction, prove that for all positive integers n, $7^n - 4^n$ is divisible by 3.*

   We start by noting that $7^1 - 4^1 = 3$, which is indeed divisible by 3, establishing our base case. Then we can note that for any $n > 1$,

   $$7^n - 4^n = 7 \cdot 7^{n-1} - 4 \cdot 4^{n-1} = 3 \cdot 7^{n-1} + 4\left(7^{n-1} - 4^{n-1}\right).$$

   Clearly, $3 \cdot 7^{n-1}$ is divisible by 3, and by our inductiion hypothesis, $7^{n-1} - 4^{n-1}$ is also divisible by 3, so $3 \cdot 7^{n-1} + 4\left(7^{n-1} - 4^{n-1}\right)$ is divisible by 3.

2. **(10 points)** *Prove combinatorially (using inclusion-exclusion) that*

   $$\binom{n}{k} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{n}{j} \binom{n+k-2j-1}{n-1}.$$

   The left side clearly counts the number of ways to select k elements out of n, or, in a balls-in-boxes paradigm, the number of ways to put k blank balls into n labeled boxes without more than one ball per box.

   To count this a different way, we could use inclusion-exclusion to count *all* distributions of k blank balls to n labeled boxes, and remove those in which some box is multiply occupied. To do this, let us denote by U the set of all distributions of k blank balls to n labeled boxes, and let A_i represent those distributions which "fail" on box i, i.e. those where box i contains at least 2 balls.

   Our balls-and-walls model will give $|U| = \binom{n+k-1}{n-1}$. We can also apply the balls-and-walls model to figure out how many elements each A_i has; pre-emptively assign 2 balls to box i, and then we have $k-2$ balls and $n-1$ walls, resulting in $|A_i| = \binom{n+k-3}{n-1}$. Of course, we pre-emptively assign 4 balls to build elements of $A_i \cap A_j$, so $|A_i \cap A_j| = \binom{n+k-5}{n-1}$, and so forth. We thus see that the j-set overlap quantity k_r is equal to $\binom{n+k-2j-1}{n-1}$, which we put into the inclusion-exclusion context to get that

   $$|U - (A_1 \cup A_2 \cup \cdots \cup A_n)| = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \binom{n+k-2j-1}{n-1}$$

   Now we could note that whenever $j > \frac{k}{2}$, it follows that $n+k-2j-1 > n-1$, so $\binom{n+k-2j-1}{n-1} = 0$, and trim off the final terms of the sum to get that one way of counting the same structure which we already counted as $\binom{n}{k}$ is with the formula $\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{n}{j} \binom{n+k-2j-1}{n-1}$.

3. **(10 points)** *Recall that a* derangement *of length n is a permutation of the numbers $\{1, \ldots, n\}$ such that no number i appears in the ith position. Let d_n represent the number of derangements of length n.*

(a) *Give a combinatorial argument to prove the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$.*

For a permutation π, let us use $\pi(i)$ to denote the number in ith position.

In a derangement π of length n, we know $\pi(n) \neq n$, so there are $n-1$ possible values for $\pi(n)$. Let us denote $\pi(n) = i$, and then consider two possible cases:

Case I: $\pi(i) = n$. Then i and n have swapped position, and the remaining $n-2$ elements of π form a derangement among themselves. This could happen in d_{n-2} different ways.

Case II: $\pi(i) \neq n$. In this case, we have the n numbers $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ being assigned to positions $\{1, 2, \ldots, n-1\}$ with one specific number being forbidden in each position: $\pi(1) \neq 1$, $\pi(2) \neq 2$, and so forth up to $\pi(i-1) \neq i-1$, with the special restriction $\pi(i) \neq n$, and then continuing with $\pi(i+1) \neq i+1$, $\pi(i+2) \neq i+2$, and so forth up to $\pi(n-1) \neq \pi(n-1)$. Since we are forbidding a specific unique value in each position of a permutation of length $n-1$, this has exactly the same count as the derangements of length $n-1$, which is d_{n-1}.

Thus, any specific choice of i yields $d_{n-1} + d_{n-2}$ derangements of length n. There are, as mentioned above, $n-1$ valid choices for i, so there are a total of $(n-1)(d_{n-1} + d_{n-2})$ derangements of length n.

(b) *Using induction on the above recurrence, prove that*

$$d_n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!}.$$

(Note that this was proved with inclusion-exclusion in class; here we are using a different method to prove the same result)

We can easily verify that $d_1 = \frac{1!}{0!} - \frac{1!}{1!} = 0$ and that $d_2 = \frac{2!}{0!} - \frac{2!}{1!} + \frac{2!}{2!} = 1$, demonstrating our base case. Now, for the inductive step, we can use the recurrence, expand the right side by invoking our inductive hypothesis, and perform a great deal of arithmetical reorganization:

$$\begin{aligned}
d_n &= (n-1)(d_{n-1} + d_{n-2}) \\
&= (n-1)\left(\sum_{k=0}^{n-1} \frac{(-1)^k (n-1)!}{k!} + \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)!}{k!}\right) \\
&= (n-1)\left(\frac{(-1)^{n-1}(n-1)!}{(n-1)!} + \sum_{k=0}^{n-2} (-1)^k \frac{(n-1)! + (n-2)!}{k!}\right) \\
&= (n-1)\left((-1)^{n-1} + \sum_{k=0}^{n-2} (-1)^k \frac{n(n-2)!}{k!}\right) \\
&= (-1)^{n-1}(n-1) + \sum_{k=0}^{n-2} (-1)^k \frac{n!}{k!} \\
&= \frac{(-1)^{n-1} n!}{(n-1)!} + \frac{(-1)^n n!}{n!} + \sum_{k=0}^{n-2} (-1)^k \frac{n!}{k!} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}
\end{aligned}$$

4. **(10 points)** *How many permutations of* $\{1, \ldots, 9\}$ *are there such that* 1 *does not immediately precede* 2*,* 2 *does not immediately precede* 3*, and so forth up to* 8 *not immediately preceding* 9*? One obvious example of such a permutation might be* 987654321*, but there are many others, such as* 132465879 *or* 351724698.

   We can use inclusion-exclusion, forbidding certain properties. Our universe of possibilities will be the set of *all* permutations of $\{1, \ldots, 9\}$. Now there are eight subsets we need to forbid: A_1, consisting of those permutations with 1 immiediately preceding 2; A_2, consisting of those where 2 precedes 3, and so forth up to A_8, consisting of those where 8 precedes 9.

   We know, of course, that $|U| = 9!$. We can find A_i by considering it as a permutation of the individual elements $1, 2, 3, \ldots, \boxed{i \quad i+1}, i+2, \ldots, 9$; that is, we'll "glue" i and $i+1$ together, and move them around collectively. So there are 8 items to be rearranged here and thus each $|A_i| = 8!$. The same trick will work on $|A_i \cap A_j|$, where we will end up either with three consecutive numbers glued together, or two individual pairs glued together, and in either case we end up with 7 movable units, resulting in 7! elements of $|A_i \cap A_j|$. This pattern continues all the way to $|A_1 \cap \cdots A_8| = 1! = 1$, which should not be at all surprising; note that at each stage introducing a new restriction into the intersection of several A_is corresponds to "gluing" two blocks together; whether the blocks are atoms or already-glued-together larger pieces is moot, because the end result is simply a reduction of 1 in the number of blocks regardless. Now we can use inclusion-exclusion to get the result

$$\sum_{j=0}^{8} (-1)^j \binom{8}{j} (9-j)! = 148329.$$

   Note that this resembles (but is not quite identical to) the formula for the number of derangements. The number of permutations of this form is approximately $\frac{(n+1)!}{ne}$, while the number of derangements is approximately $\frac{n!}{e}$.

So, naturalists observe, a flea
Has smaller fleas that on him prey,
And these have smaller still to bite 'em,
And so proceed *ad infinitum*. —Jonathan Swift, "On Poetry: A Rhapsody"