1. **(10 points)** Using induction, prove that for all positive integers n, $7^n - 4^n$ is divisible by 3.

2. **(10 points)** Prove combinatorially (using inclusion-exclusion) that

$$\binom{n}{k} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^j \binom{n}{j} \binom{n+k-2j-1}{n-1}.$$

3. **(10 points)** Recall that a *derangement* of length n is a permutation of the numbers $\{1, \ldots, n\}$ such that no number i appears in the ith position. Let d_n represent the number of derangements of length n.

   (a) Give a combinatorial argument to prove the recurrence $d_n = (n-1)(d_{n-1} + d_{n-2})$.

   (b) Using induction on the above recurrence, prove that

$$d_n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!}.$$

   (Note that this was proved with inclusion-exclusion in class; here we are using a different method to prove the same result)

4. **(10 points)** How many permutations of $\{1, \ldots, 9\}$ are there such that 1 does not immediately precede 2, 2 does not immediately precede 3, and so forth up to 8 not immediately preceding 9? One obvious example of such a permutation might be 987654321, but there are many others, such as 132465879 or 351724698.

So, naturalists observe, a flea
Has smaller fleas that on him prey,
And these have smaller still to bite 'em,
And so proceed *ad infinitum*. —Jonathan Swift, "On Poetry: A Rhapsody"