1. **(10 points)** *Let $\mathcal{S}$ be a collection of subsets of $\{1, \ldots, n\}$.*

   (a) **(4 points)** *Demonstrate that it is possible for $|\mathcal{S}|$ (i.e., the number of* subsets *of $\{1, \ldots, n\}$ contained as* elements *of $\mathcal{S}$) to equal 2^{n-1} without any two elements of $\mathcal{S}$ being disjoint from each other.*

   Let's consider all the subsets which contain 1, i.e., let's take every subset of $\{2, 3, \ldots, n\}$ and insert 1 into each. This is a collection of 2^{n-1} sets none of which are mutually disjoin, since for every pair of sets A and B, the two sets have at least one element (i.e., the element 1) in common.

   (b) **(6 points)** *Prove that if $|\mathcal{S}| \geq 2^{n-1} + 1$, then $\mathcal{S}$ contains two elements which are disjoint from each other.*

   Let us create pigeonholes for the subsets of $\{1, 2, \ldots, n\}$, with each pigeonhole being home to exactly two sets: a set S and its complement $\{1, 2, \ldots, n\} - S$. Since there are 2^n subsets of $\{1, 2, \ldots, n\}$ in total, this partition results in $\frac{2^n}{2} = 2^{n-1}$ pigeonholes. Thus, if we have a collection of $2^{n-1} + S$ subsets of $\{1, \ldots, n\}$, then by the Pgeonhole Principle two of them must be in the same pigeonhole, so by the scheme used to construct our pigeonholes, those two are complementary to each other and thus disjoint.

2. **(10 points)** *Suppose that the sequence of numbers a_n is given by the recurrence $a_n = 4a_{n-1} + n$ with initial value $a_1 = 0$. Prove that $a_n = \Theta(4^n)$.*

Just to get a feel for this sequence, it's illuminating to work out the first several elements: $a_1 = 0$, $a_2 = 2$, $a_3 = 11$, $a_4 = 48$, $a_5 = 197$, $a_6 = 794$. We can see that all of these are well below 4^n. Also, let's note that proving that this is $\Theta(4^n)$ includes, as subordinate arguments, neeeding to show that it is both $O(4^n)$ and $\Omega(4^n)$.

Proving that $a_n = \Omega(4^n)$ is easy; let's simply show by induction that for $n \geq 2$, $a_n \geq \frac{1}{8}4^n$. This is clearly true in the base case $n = 2$, since $a_2 = 2 = \frac{16}{8}$. Now let us assume that $a_n \geq \frac{1}{8}4^n$ and seek to show a lower bound on a_{n+1}, making use of our recurrence:

$$a_{n+1} = 4a_n + (n+1) > 4a_n \geq 4\left(\frac{1}{8}4^n\right) = \frac{1}{8}4^{n+1}$$

completing our inductive proof.

On the other hand, showing $a_n = O(4^n)$ is a bit harder, because that annoying addition causes our ratio with a^n to creep upwards, although in a bounded fashion, so what we basically have to prove is that an upper bound on a_n is some $f(n)4^n$, where $f(n)$ is itself bounded above by a constant. Since we can be ridiculously liberal with our bound, let's try showing by induction that $a_n \leq \left(1 - \frac{1}{n}\right)4^n$. This is true and in fact sharp for $n = 1$, demonstrating our base case. For larger values, let us assume $a_n \leq \left(1 - \frac{1}{n}\right)4^n$ and seek to place an upper bound on a_{n+1} using our recurrence:

$$a_{n+1} = 4a_n + (n+1) \leq 4\left(1 - \frac{1}{n}\right)4^n + (n+1) = \left(1 - \frac{1}{n} + \frac{n+1}{4^{n+1}}\right)4^{n+1} \leq \left(1 - \frac{1}{n+1}\right)4^{n+1}$$

The last inequality in the line above is not wholly obvious; it is essentially the assertion that $\frac{1}{n} - \frac{n+1}{4^{n+1}} \geq \frac{1}{n+1}$. Multiplying by a common denominator and rearranging, however, it is the inarguably true inequality $4^{n+1} \geq n^3 + 2n^2 + n$.

3. **(10 points)** *If you are given an unsorted list of n numbers, how long will it take to determine whether there are three numbers x, y, and z in the list such that $xy = z$? Give your answer in big-O notation and explain your reasoning.*

   A simple algorithm to do this is to iterate x over the entire length of the list, y over the entire length of the list, and z over the entire length of the list, checking for each triple whether $xy = z$; if we find values for which $xy = z$, we can answer "yes"; otherwise after we completely exhaust every possibility we can say "no". But this involves looking at *every* triple (x, y, z) in the list (in practice we could halve the sample space, since (x, y, z) and (y, x, z) represent the same proposition), so it would take $O(n^3)$ time to complete this process.

   That was all I expected—and, to be honest, all I came up with—but props to Sean Harper and Kelsey Hough for coming up with an approach which uses slightly more space but less time: we can sort the list in $O(n \log n)$ time (or $O(n^2)$ time using a less efficient sort), and then, for each choice of the pair (x, y), perform a binary search to see if the product xy is in the list. The binary search itself takes $O(\log n)$ time, and is performed on $O(n^2)$ pairs (x, y), so this search will take $O(n^2 \log n)$ time, which subsumes the $O(n \log n)$ or $O(n^2)$ time spent by the sort, so this algorithm completes in $O(n^2 \log n)$ time.

4. **(10 points)** *Ten people go to a party and are introduced; everyone shakes hands with at least one other person. Prove that there must be at least two partygoers who shook hands with the same number of people. Is this still true if not everyone shakes hands with at least one person?*

   Each person shook hands with between 1 and 9 other people; if we classify people by the number of hands they shook, we are thus classifying them into 9 pigeonholes, and since there are 10 pigeons, two are in the same hole so that two people shook the same number of hands.

   If, on the other hand, someone could have shaken 0 hands, then there is a plausible assignation of pigeons to pigeonholes without collision, since now our pigeonholes are numbered from 0 to 9, and there are 10 of them. However, despite the numerical plausibility of such a scenario, it cannot occur, for two reasons.

   One simple argument why that would be impossible is because one person at the party would have shaken 9 hands, i.e., been introduced to every other person, while one other person would have shaken 0 hands, i.e., not been introduced to any other person. These two requirements would be contradictory, since the extremely sociable and extremely unsociable person would have to simultaneously meet and not meet.

   Alternatively, in order for this to happen, then when we add up the number of handshakes each person has taken part in, we get $0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, and this will turn out to be impossible. It is impossible to get such a total because each handshake contributes 2 to the total (since each of the two handshake participants would count it), and so this total should always be even, which 45 is not. This alternative approach has the distinct disadvantage that it only results in a contradiction when the number of people is 2 or 3 modulo 4, whereas the argument in the previous paragraph always works.

On two occasions I have been asked — "Pray, Mr. Babbage, if you put into the machine wrong figures, will the right answers come out?" In one case a member of the Upper, and in the other a member of the Lower House put this question. I am not able rightly to apprehend the kind of confusion of ideas that could provoke such a question.
—Charles Babbage