1. **(10 points)** *Let a_n represent the number of ways to distribute n blank balls to three labeled boxes such that the first box contains an even number of balls, the second contains an odd number of balls, and the third contains at least 2 balls.*

   (a) *Find a closed form for the generating function $f(x) = \sum_{n=0}^{\infty} a_n x^n$.*
   The generating function associated with the assignment of balls to the first box is $1 + x^2 + x^4 + \cdots = \frac{1}{1-x^2}$; the generating function associated with the second box is $x + x^3 + x^5 + \cdots = \frac{x}{1-x^2}$, and the generating function for the third box is $x^2 + x^3 + x^4 + \cdots = \frac{x^2}{1-x}$; their product is thus

   $$f(x) = \frac{x^3}{(1-x^2)^2(1-x)} = \frac{x^3}{(1-x)^3(1+x)^2}$$

   (b) *Decompose this generating function into partial fractions (a calculator or equation-solving system may help).*
   The partial fraction decomposition involves 5 possible terms, matching the prototype

   $$\frac{x^3}{(1-x)^3(1+x)^2} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1+x} + \frac{E}{(1+x)^2}$$

   which, when multiplying by the common denominator, yields:

   $$x^3 = A(1-x)^2(1+x)^2 + B(1-x)(1+x)^2 + C(1+x)^2 + D(1-x)^3(1+x) + E(1-x)^3$$

   We could solve a system of five equations in five unknowns to determine this, but it is slightly less painful to begin by using the Heaviside method: plugging in $x = 1$ we get $1 = 4C$, and plugging in $x = -1$ we get $-1 = 8E$, so $C = \frac{1}{4}$ and $E = \frac{-1}{8}$. Now we have only three unknowns:

   $$x^3 = A(1-2x^2+x^4) + B(1+x-x^2-x^3) + \frac{1+2x+x^2}{4} + D(1-2x+2x^3-x^4) - \frac{1-3x+3x^2-x^3}{8}$$

   $$-\frac{1}{8} - \frac{7}{8}x + \frac{1}{8}x^2 + \frac{7}{8}x^3 = A(1-2x^2+x^4) + B(1+x-x^2-x^3) + D(1-2x+2x^3-x^4)$$

   And considered termwise, we see that

   $$\begin{cases} \frac{-1}{8} = A + B + D \\ \frac{-7}{8} = B - 2D \\ \frac{1}{8} = -2A - B \\ \frac{7}{8} = -B + 2D \\ 0 = A - D \end{cases}$$

From which it is easy to determine that $A = D$ and then, adding the first and fourth equations, that $A + 3D = \frac{3}{4}$, so $A = \frac{3}{16}$, $D = \frac{3}{16}$ and $B = 2D - \frac{7}{8} = \frac{-1}{2}$. Thus:

$$f(x) = \frac{1}{16}\left[\frac{3}{1-x} - \frac{8}{(1-x)^2} + \frac{4}{(1-x)^3} + \frac{3}{1+x} - \frac{2}{(1+x)^2}\right].$$

(c) *Using your partial fraction decomposition, determine a formula for a_n.*

We shall expand terms in our partial fraction decomposition to get a power series representation, whose coefficient will be a_n:

$$\begin{aligned} f(x) &= \frac{1}{16}\left[\frac{3}{1-x} - \frac{8}{(1-x)^2} + \frac{4}{(1-x)^3} + \frac{3}{1+x} - \frac{2}{(1+x)^2}\right] \\ &= \frac{1}{16}\left[3\sum_{n=0}^{\infty}\binom{n}{0}x^n - 8\sum_{n=0}^{\infty}\binom{n+1}{1}x^n + 4\sum_{n=0}^{\infty}\binom{n+2}{2}x^n + 3\sum_{n=0}^{\infty}\binom{n}{0}(-x)^n - 2\sum_{n=0}^{\infty}\binom{n+}{1} \right. \\ &= \sum_{n=0}^{\infty}\frac{3}{16}x^n - \sum_{n=0}^{\infty}\frac{8}{16}(n+1)x^n + \sum_{n=0}^{\infty}\frac{2}{16}(n+2)(n+1)x^n + \sum_{n=0}^{\infty}\frac{3}{16}(-1)^n x^n - \sum_{n=0}^{\infty}\frac{2}{16}(n+1) \\ &= \sum_{n=0}^{\infty}\frac{-1-2n+2n^2+(1-2n)(-1)^n}{16}x^n \end{aligned}$$

so $a_n = \frac{-1-2n+2n^2+(1-2n)(-1)^n}{16}$, which, notably, is in fact always an integer (since it is in fact counting something).

2. **(10 points)** *Let a_n be the coefficient on x^n in the power-series expansion of $f(x) = \frac{(1+x^2+x^4)x^2}{(1-x)^3(1-x^3)(1-x^{10})}$ (or, equivalently, you could let $a_n = n!f^{(n)}(0)$, using the Maclaurin series). Describe a combinatorial question to which a_n is the answer (i.e., "there are a_n ways to perform the following process…").*

This power series would reflect a distribution process which logically splits into 6 parts, so we might be talking about distributions of n identical objects to six recipients, with each recipient's rule dictating one part of the given function. So, for instance, the factor $(1+x^2+x^4)$ might describe recipient A only being allowed to receive 0, 2, or 4 objects; the factor $\frac{x^2}{1-x} = x^2+x^3+x^4+\cdots$ might describe recipient B being required to receive at least 2 objects; the two factors $\frac{1}{1-x} = 1+x+x^2+\cdots$ would describe recipeints C and D having no restrictions on what they get; the factor $\frac{1}{1-x^3} = 1+x^3+x^6+x^9+\cdots$ would require recipient E to receive a multiple of 3 objects, and the factor $\frac{1}{1-x^{10}} = 1+x^{10}+x^{20}+\cdots$ would require recipient F to receive a multiple of 10 objects.

3. **(10 points)** *Explore the conjugates of the partitions of n into distinct parts. What property defines these partitions, and what is the generating function for the number of partitions with this property?*

The second question is actually the easier one here; the partitions being looked at are clearly equinumerous with the partitions of n into distinct parts, which it is easy to build a generating function for: $(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$. If you want to define a generating function based on the specific property we discover in the question, though, read on!

As to the first question, we might consider describing the property of having distinct parts geometrically in terms of the Ferrers diagram as having each row of different length, so that row lengths are strictly decreasing. Then, on conjugation, we will find that column lengths (from left to right) are strictly decreasing. Note that two columns of the same length would correspond to a row two units longer than its successor, so in order to avoid this each row will have to be no more than one unit longer than its successor. In other words, we want row lengths (i.e., elements of the partition) to consist of at least one instance of consecutive sizes, starting with 1.

As an illustration, let us consider the distinct partitions of 8: 8, $7+1$, $6+2$, $5+3$, $5+2+1$, and $4+3+1$. These have as their conjugates $1+1+1+1+1+1+1+1$, $2+1+1+1+1+1+1$, $2+2+1+1+1+1$, $2+2+2+1+1$, $3+2+1+1+1$, and $3+2+2+1$. Note that each of these consists either solely of 1s, consists of at least one 1 and at least one 2, or consists of at least one 1, 2, and 3, as the above argument suggests should be the case.

If we want a generating function based on this, then we could sum over a range of values for k, the size of the largest part, and then multiply together selection procedures for the number of parts of each size up to k, ensuring that each part appears at least once:

$$\sum_{k=0}^{\infty}(x+x^2+x^3+\cdots)(x^2+x^4+x^6+\cdots)(x^3+x^6+x^9+\cdots)\cdots(x^k+x^{2k}+x^{3k}+\cdots)$$

$$= \sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{x^i}{1-x^i} = \sum_{k=0}^{\infty}\frac{x^{k(k+1)/2}}{\prod_{i=1}^{k}(1-x^i)}$$

That this function is identical to the generating function for distinct partitions, $\prod_{i=1}^{\infty}(1+x^i)$, and to the generating function for odd partitions, $\prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}$, is not actually that easy to show algebraically.

4. **(10 points)** *Let a_n be the number of n-letter strings consisting of the letters A, B, C, and D with at least one A and an even number of Cs.*

   (a) *Find a formula for the exponential generating function $\sum_{n=0}^{\infty}\frac{a_n}{n!}x^n$.*

   The generating functions associated with the inclusion of As is $x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x-1$; both B and D are associated with the generating function $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x$, while C has generating function $1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots = \cosh(x) = \frac{e^x+e^{-x}}{2}$. Thus this expression as a whole has generating function

   $$f(x) = (e^x-1)(e^x)^2\frac{e^x+e^{-x}}{2} = \frac{e^{4x}-e^{3x}+e^{2x}-e^x}{2}$$

   (b) *Using the above generating function, find a closed formula for a_n.*

Note that, using the known power series for e^u:

$$f(x) = \sum_{n=0}^{\infty} \frac{\frac{(4x)^n}{n!} - \frac{(3x)^n}{n!} + \frac{(2x)^n}{n!} - \frac{x^n}{n!}}{2} = \sum_{n=0}^{\infty} \frac{4^n - 3^n + 2^n - 1}{2} \frac{x^n}{n!}$$

so $a_n = \frac{4^n - 3^n + 2^n - 1}{2}$.

A matematikus olyan gép, amely kávéból tételeket gyárt. [A mathematician is a device for converting coffee into theorems.] —Alfréd Rényi