1. **(10 points)** *You open an account at a bank that pays 5% interest yearly, and deposit a_0 dollars in it. Every year you withdraw \$10 times the number of years you have had the account. For example, if you started with \$1000, then in the first year you would earn \$50 in interest and withdraw \$10, leaving \$1040, and in the second year would earn \$52 and withdraw \$20, leaving \$1072, and so forth.*

   (a) Find a recurrence for a_n, the balance in the account after n years.

   Since the balance after n years is 105% of the previous year's balance, minus $\$10n$, *we have* $a_n = 1.05a_{n-1} - 10n$.

   (b) Solve the recurrence to find a closed form for a_n.

   The associated homogeneous recurrence $b_n = 1.05b_{n-1}$ *clearly has general solution* $b_n = A1.05^n$ *for undetermined coefficient A. Now we shall find a particular solution to the inhomogeneous equation, using the template* $a_n^p = Bn+C$. *Substituting this particular solution into the recurrence, we find that*

$$\begin{aligned} Bn + C &= 1.05(B(n-1) + C) - 10n \\ 10n &= 0.05Bn + (0.05C - 1.05B) \end{aligned}$$

   so that $B = 200$ *and* $C = 4200$, *yielding a particular solution* $a_n^p = 200n + 4200$. *The general solution is thus* $a_n = A1.05^n + 200n + 4200$. *Plugging in a particular value of* a_0, *we see that* $a_0 = A + 4200$, *so* $A = a_0 - 4200$, *leading to the solution*

$$a_n = (a_0 - 4200)1.05^n + 200n + 4200.$$

   (c) What is the smallest initial deposit which would guarantee that the account never runs out of money?

   Over the long term, the balance will be positive or negative based on the coefficient of 1.05^n *in the closed form, since this is the dominant term. Thus, in order for the bank balance to remain non-negative (and actually positive, as it turns out) indefinitely, it must be the case that* $a_0 - 4200 \geq 0$, *so* $a_0 \geq 4200$. *Thus, an initial deposit of \$4200 is necessary.*

2. **(10 points)** *Let* $a_n = 8a_{n-1} - 16a_{n-2} + 3 \cdot 4^n$ *with* $a_0 = 3$ *and* $a_1 = -1$. *Find a closed form for* a_n.

   The associated homogeneous recurrence, $b_n = 8b_{n-1} - 16b_{n-2}$, has characteristic polynomial $\lambda^2 = 8\lambda - 16$, which has a double root at $\lambda = 4$. Thus, the general solution to the homogeneous recurrence is $b_n = A4^n + Bn4^n$.

   Now we shall find a particular solution to the inhomogeneous equation. If we were to work in ignorance of our homogeneous solution, we would use the template $a_n^p = B4^n$, but must "bump it upwards" to not overlap our homogeneous solution, and will instead consider $a_n^p = Bn^2 4^n$. Then, substituting this into our recurrence:

$$\begin{aligned}
Bn^2 4^n &= 8B(n-1)^2 4^{n-1} - 16B(n-2)^2 4^{n-2} + 3 \cdot 4^n \\
16Bn^2 4^{n-2} &= 32B(n^2 - 2n + 1)4^{n-2} - 16B(n^2 - 4n + 4)4^{n-2} + 48 \cdot 4^{n-2} \\
16Bn^2 &= 32Bn^2 - 64Bn + 32B - 16Bn^2 + 64Bn - 64B + 48 \\
32B &= 48 \\
B &= \frac{3}{2}
\end{aligned}$$

so $a_n^p = \frac{3}{2}n^2 4^n$, leading to the general form $a_n = (A + Bn + \frac{3}{2}n^2)4^n$. We substitute this into our initial conditions to get:

$$\begin{cases} 3 = a_0 = A \\ -1 = a_1 = 4A + 4B + 6 \end{cases}$$

so that $A = 3$ and $B = \frac{-19}{4}$, leading to the solution

$$a_n = \left(3 - \frac{19}{4}n + \frac{3}{2}n^2\right) 4^n.$$

3. **(10 points)** *We are making bracelets with 6 stones in a ring, with three different colors of stone. A bracelet must contain at least one stone of each color. Two bracelets are considered to be identical if one is simply a rotation or a flip of the other. How many different bracelets are possible?*

   The symmetries we want to be considering are those of the dihedral group on 6 elements, which contains 12 elements: the identity, the clockwise and counterclockwise 60° rotations r and r^5, the clockwise and counterclockwise 120° rotations r^2 and r^4, and the 180° rotation r^3, as well as the rotations around antipodal stones f, r^2f, and r^4f, as well as the rotations around antipodal edges between stones rf, r^3f, and r^5f.

   All possible oriented bracelets are invariant under the identity, so the number of invariants under the identity is simply the ordered selection of six elements form a pool of three possibilities, with each needing to be selected at least once; this is, as we have seen with the twelvefold way, simply $3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 540$.

   Now, we might note that no bracelets are invariant under r or r^5; such bracelets would need to all be a single color, violating the condition that bracelets have stones of all three colors. Likewise, r^2 or r^4 would necessitate bracelets which have at most two colors in an alternating ring, so there are also no invariants for these.

   Invariance under r^3 requires that the bracelet be made up of two repetitions of a single pattern of three stones, which is possible, and is equal to the number of ways to arrange all three stones, which is 6 (alternatively, we could look at number of ways to select ordered stones using each, which would be $3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 = 6$.

   Invariance under rf, r^3f, and r^5f is quite similar, in that the bracelet is defined by the colors on three specific stones, and these too have 6 invariants. Finally, f, rf^2, and

rf^4 involve fixing two stones and mapping the other two pairs onto each other, so the bracelet is defined by a selection of 4 stones, in $3^4 - 3 \cdot 2^4 + 3 \cdot 1^4 = 36$ ways.

Now we may use Burnside's Lemma to count the total number of possible bracelets:

$$\frac{540 + 0 + 0 + 6 + 0 + 0 + 36 + 6 + 36 + 6 + 36 + 6}{12} = 56$$

4. **(10 points)** *A 4×4 grid of squares is filled in, with each of the 16 squares colored black or white. Two colorings are regarded as identical if one can be converted to each other by performing any combination of flipping, rotating, or swapping the two colors (flipping all the black squares to white and vice versa). How many non-identical colorings are there?*

   The algebra here is slightly more complicated than simply the eight-element group D_4; each of those eight elements could be composed with a "color-swap" to get 16 possible elements; if we denote the color-swap as s, then these elements could be called e, r, r^2, r^3, f, rf, r^2f, r^3f, s, rs, r^2s, r^3s, fs, rfs, r^2fs, and r^3fs. Every single coloring of the 4×4 square is invariant under the identity, for a total of $2^{16} = 65536$ invariant colorings. To be invariant under r or r^3, a coloring would have to have the same 2×2 block repeated (with appropriate rotations) on all four corners, so there are $2^4 = 16$ invariants. Invariance under r^2 depends on one half of the grid being identical to the appearance on the other half, so there are $2^8 = 256$ invariants. Likewise, letting f be a horizontal flip, invariants under both f and r^2f are determined by the coloration of half (either a vertical or horizontal half) of the grid, so there are $2^8 = 256$ invariants there. rf and r^3f are corner-to-corner flips, so they have free choice of color for the four squares along their diagonals and then for each of the 6 pairs of mirror-image squares off the diagonal, for a total of 10 choices of colors and thus $2^{10} = 1024$ invariant colorings.

   Looking at the transformations involving s, the exact same analysis, simply with a color-transformation added to it, works for most of the transformations; i.e. r^2s has 2^8 invariants, fs has 2^8, rs has 2^4, etc. However, all of the swapped versions of transformations which leave a square unmoved (s, rfs, and r^3fs) have *no* invariants, since the unmoved square is necessarily a different color after transformation. Thus, applying Burnside's lemma gives:

$$\frac{65536 + 16 + 256 + 16 + 256 + 1024 + 256 + 1024 + 0 + 16 + 256 + 16 + 256 + 0 + 256 + 0}{16} = 4324.$$

> Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now, as in the past, are inspired by the art of mathematics rather than by any prospect of ultimate usefulness.
> —Eric Temple Bell