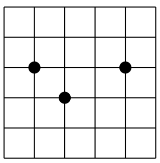
1. **(10 points)** *Prove by induction that for every integer $n \geq 1$, it is the case that $2 \cdot n! \geq 2^n$.*

   For the base step, let us note that $2 \cdot 1! = 2 \geq 2^1$, so the above statement is clearly true for $n = 1$. Now, given the inductive hypothesis $2 \cdot n! \geq 2^n$, we wish to show that $2 \cdot (n+1)! \geq 2^{n+1}$. Noting that $n + 1$ is at least 2, we can then multiply both sides by $n + 1$ to get $2 \cdot n! \cdot (n+1) \geq 2^n(n+1) \geq 2^n \cdot 2 = 2^{n+1}$. The left side can be simplified to $2(n+1)!$¡ so we get that $2(n+1)! \geq 2^{n+1}$.

2. **(10 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following grid which* pass through at least one of the marked points*?*

   Let A, B, and C be the sets of paths through $(2,2)$, $(1,3)$, and $(4,3)$ respectively. We want to find $|A \cup B \cup C|$ here, and will use inclusion-exclusion. Note that $|A| = \binom{4}{2}\binom{6}{3} = 120$, $|B| = \binom{4}{1}\binom{6}{2} = 60$, and $|C| = \binom{7}{3}\binom{3}{1} = 105$. Looking at intersections, $|A \cap B| = 0$, since no path goes through both $(2,2)$ and $(1,3)$, while $|A \cap C| = \binom{4}{2}\binom{3}{1}\binom{3}{1} = 54$ and $|B \cap C| = \binom{4}{1}\binom{3}{0}\binom{3}{1} = 12$. Thus:

   $$|A \cup B \cup C| = 120 + 60 + 105 - 0 - 54 - 12 + 0 = 219.$$

3. **(10 points)** *Show via a combinatorial proof that for $0 \leq k \leq n$,*

   $$\sum_{j=k}^{n} \binom{n}{j}\binom{j}{k} = \binom{n}{k} 2^{n-k}.$$

   The right side counts what is clearly a selection in two parts: a "special subset" of k elements of $\{1, 2, 3, \ldots, n\}$, together with a subset built from any number of the remaining $n-k$ elements; that is to say, on the right side we construct a pair of *disjoint* subsets (A, B) of $\{1, \ldots, n\}$, with $|A| = k$. On the left side, however, we construct a set of at least k elements (specifically, of j elements, with j ranging from k to n), and then construct a subset of exactly k elements therefrom. So on the left side we're building pairs (R, S) where $R \supseteq S$, and $|S| = k$. We can easily relate these two constructions to each other by letting $A = S$ and $R = A \cup B$, or, alternatively, letting $B = R - S$.

   A more folksy explanation might be that this describes the number of ways of recruiting for an organization with a k-person board of directors but otherwise of no fixed size from a population of n people. The left and right sides describe two different processes for achieving this result: the three symbols on the left represent, in order, deciding how large the organization will be (denoting it j, and permitting any size from k on up), which members of the population will join (there are $\binom{n}{j}$ possibilities), and then, from among the just-selected members of the organization, further selecting k of them to be on the board of directors (in any of $\binom{j}{k}$ ways); on the right, we instead first select the board from the general population (in any of $\binom{n}{k}$ ways) and, for each of the remaining $n-k$ members of the population, make a binary choice as to whether they are in the organization or not.

4. **(5 points)** *What is the coefficient of x^2 in the expansion of $(3x+4)^6$?*

We know that the binomial expansion yields the term $\binom{6}{2}(3x)^2(4)^4$, so segregating out the constants yields $3^2 4^4 \binom{6}{2} x^2 = 34560x^2$ so the coefficient is 34560.

5. **(25 points)** *A far-future dystopia is replacing everyone's names with 7-digit codes using only the digits 1, 2, 3, and 4.*

   (a) **(5 points)** *How many different "names" are possible under this scheme?*

   Each digit of someone's name could be any of 4 different values, and a name is made up of 7 such choices, so there are $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^7 = 16384$ names.

   (b) **(10 points)** *Two people are said to be in the same artificial family if their names are anagrams of each other; for instance, citizens 1314221 and 2211134 are both in the same family. How many different families are there?*

   A "family" here is just a set of digit-frequencies; for instance, the above family is a three-ones, two-twos, single-three-and-four family. Thus, a family can be represented by the distribution of seven *unlabeled* balls to four labeled boxes, which is best modeled with a balls-and-walls paradigm of seven balls and three walls, which can be distributed in $\binom{10}{3} = 105$ ways for a total of 105 families.

   Note that families come in widely varying sizes. Four families (the "all-1s", "all-2s", "all-3s", and "all-4s") have a single member each, while four families (each with a 2+2+2+1 distribution of name digits) would have $\binom{7}{2,2,2,1} = 630$ members. The *average* family size would be $\frac{4^7}{\binom{10}{3}} \approx 156.04$, which is of course not an integer but there is no good reason to expect an average to be an integer.

   (c) **(10 points)** *A citizen is considered to be an "elite" if their name contains every digit at least once. How many elite names are there?*

   This is a classic surjective-distribution problem, solvable via inclusion-exclusion; taking all 4^7 names, we wish to exclude from consideration those which lack a 1, lack a 2, lack a 3, or lack a 4. These exclusions each have 3^7 elements; their pairwise intersections each have 2^7 elements, and their triplewise intersections each have 1^7 elements (and the quadruplewise intersection is empty), so by inclusion-exclusion, there are

$$4^7 - \binom{4}{1}3^7 + \binom{4}{2}2^7 - \binom{4}{3}1^7 + \binom{4}{4}0^7 = 8400$$

   elites (who belong, incidentally, to $\binom{7-1}{4-1} = 20$ families, which range in size from $\binom{7}{4,1,1,1} = 210$ members to $\binom{7}{2,2,2,1} = 630$ members).

6. **(10 points)** *The Powerball is a popular lottery whose tickets (and drawings) are of five distinct numbers between 1 and 59, and a single "Powerball" number between 1 and 35 (not necessarily distinct from the five regular numbers). Among the five standard numbers, order does not matter. A ticket is a \$7 winner if it either matches* exactly 3 *of the regular numbers and not the Powerball, or if it matches* exactly 2 *of the regular numbers as well as the Powerball. The most recent drawing had the winning numbers 21, 39, 40, 55, 59, and a Powerball of 17 (note: this information is actually not necessary to answer the question). How many possible \$7-winning tickets were there on this drawing?*

A "match 3" ticket could match any 3 of the 5 winning numbers, and then have 2 non-winning numbers, and finally, any of the 34 non-winning powerballs: this can happen in

$\binom{5}{3}\binom{54}{2} \cdot 34 = 486540$ ways; a "match 2 with powerball" could match any 2 of the 5 winning numbers, 3 non-winning numbers, and the unique winning powerball: this can happin in $\binom{5}{2}\binom{54}{3} \cdot 1 = 248040$ ways, so there are a total of $486540 + 248040 = 734580$ tickets worth \$7.

7. **(10 points)** *Galangal is a ginger-like rhizome used in Southeast Asian cuisine. Answer the following questions about anagrams of the word "GALANGAL".*

   (a) **(5 points)** *How many anagrams are there in total?*

   There are 2 Ls, 2 Gs, 3 Ls, and a single N, so a multinomial coefficient is an effective measure: $\binom{8}{2,2,3,1} = \frac{8!}{2!2!3!1!} = 1680$.

   (b) **(5 points)** *How many anagrams are there which do not contain either a double G or a double L (i.e. "GLAANLGA" is OK, "LANGGLAA" is not).*

   From the above count of anagrams, we wish to subtract off those which use a double G or a double L; in addition, we will want to add back in those which use both, for inclusion-exclusion purposes.

   To count these forbiddences, we could consider the possibility of "gluing" the Gs or Ls together. So now we want anagrams of two Gs, three As, an N, and an LL, which can be made in any of $\binom{7}{2,3,1,1} = 420$ ways. In total, we will end up with:

$$\binom{8}{3,2,2,1} - \binom{7}{3,2,1,1} - \binom{7}{3,2,1,1} + \binom{6}{3,1,1,1} = 960.$$

   (c) **(5 point bonus)** *What is the probability that an anagram of "GALANGAL", chosen uniformly at random from among all the possibilities, has at least two "A"s next to each other (use the back of this sheet for work)?*

   The positions of the Gs, Ls, and Ns are irrelevant; we could introduce them, if we wanted, but it would simply add a multiplicative factor of $\binom{5}{2,2,1}$ on top and bottom and have no effect on the probability. What we thus really want to know is how many of the placements of three As have at least 2 next to each other. Note that our entire space of possibilities is $\binom{8}{3} = 56$, which is small enough that brute force might very well be a feasible approach to this problem. However, instead, we might ask which placements have an "AA" and another "A", and how many have an "A" and then another "AA". Note that this would double-count the 6 placements with "AAA", so we would have to subtract those out. Among 7 positions (counting "AA" as a single unit), we would want to place the AA and the A, which can be done in $7 \cdot 6 = 42$ ways, so there are a total of $42 - 6 = 36$ placements with at least two As next to each other. Thus the probability is $\frac{36}{56} = \frac{9}{14}$.

   Alternatively, we could count the distributions which don't place any As next to each other, and subtract from the total of 56. To do so, we might notice that the placement of 3 nonadjacent As in 8 positions can be bijectively mapped to free placement of As in 6 positions simply by removing a non-A between each two As; i.e. we can map AXXAXAXX to AXAAXX and back again. Thus there are $\binom{6}{3} = 20$ non-adjacent A locations, leading again to $56 - 36$ adjacent placements.