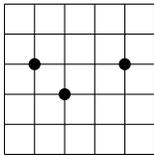


For full credit show all of your work (legibly!), unless otherwise specified. Answers need not be completely reduced unless otherwise stated — and for many questions, the calculation will be unpleasantly burdensome — and may be left in terms of sums, differences, products, quotients, factorials, and binomial coefficients.

1. **(10 points)** Prove by induction that for every integer  $n \geq 1$ , it is the case that  $2 \cdot n! \geq 2^n$ .

2. **(10 points)** How many direct paths are there from the lower left corner to the upper right corner of the following grid which *pass through at least one of the marked points*?



3. **(10 points)** Show via a combinatorial proof that for  $0 \leq k \leq n$ ,

$$\sum_{j=k}^n \binom{n}{j} \binom{j}{k} = \binom{n}{k} 2^{n-k}.$$

4. **(5 points)** What is the coefficient of  $x^2$  in the expansion of  $(3x + 4)^6$ ?
5. **(25 points)** A far-future dystopia is replacing everyone's names with 7-digit codes using only the digits 1, 2, 3, and 4.
- (a) **(5 points)** How many different "names" are possible under this scheme?
- (b) **(10 points)** Two people are said to be in the same artificial family if their names are anagrams of each other; for instance, citizens 1314221 and 2211134 are both in the same family. How many different families are there?
- (c) **(10 points)** A citizen is considered to be an "elite" if their name contains every digit at least once. How many elite names are there?

6. **(10 points)** The Powerball is a popular lottery whose tickets (and drawings) are of five distinct numbers between 1 and 59, and a single “Powerball” number between 1 and 35 (not necessarily distinct from the five regular numbers). Among the five standard numbers, order does not matter. A ticket is a \$7 winner if it either matches *exactly 3* of the regular numbers and not the Powerball, or if it matches *exactly 2* of the regular numbers as well as the Powerball. The most recent drawing had the winning numbers 21, 39, 40, 55, 59, and a Powerball of 17 (note: this information is actually not necessary to answer the question). How many possible \$7-winning tickets were there on this drawing?
7. **(10 points)** Galangal is a ginger-like rhizome used in Southeast Asian cuisine. Answer the following questions about anagrams of the word “GALANGAL”.
- (a) **(5 points)** How many anagrams are there in total?
- (b) **(5 points)** How many anagrams are there which do not contain either a double G or a double L (i.e. “GLAANLGA” is OK, “LANGGLAA” is not).
- (c) **(5 point bonus)** What is the probability that an anagram of “GALANGAL”, chosen uniformly at random from among all the possibilities, has at least two “A”s next to each other (use the back of this sheet for work)?