1. **(10 points)** *Let a_n be the number of ways to write n as a sum of positive integers in which each integer appears* at most twice. *For example, $a_4 = 4$, because there are the four partitions 4, $3+1$, $2+2$, and $2+1+1$ (but* not $1+1+1+1$*). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.*

   The generating function for selection of some quantity of ones to use is $1+x+x^2$, because no more than two ones can be used. Likewise, the generating function for selecting twos would be $1+x^2+x^3$; sdelecting threes would be done by $1+x^3+x^6$, and so forth with the pattern continuing until we see that the final generating function is

$$(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots = \prod_{i=0}^{\infty}(1+x^i+x^{2i})$$

2. **(15 points)** *Your company makes reversible picture frames, all the same size. These frames are rectangular but* not *square, and each frame has either a curlicue or no decoration on each of the two longest sides, as well as having the segments of moulding (the four timbers on the outside of the frame) each painted either red, white, or blue. One such frame (in a landscape orientation) might have a curlicue on bottom but not on top, while the bottom and top mouldings are blue, and the left and right mouldings are red and white respectively. Two frames are identical if one can be rotated* or reflected *onto the other, and you produce every possible frame (including boring ones, like an all-white frame without decorations). How many different frames are there? (Note: you may leave your result as an unsimplified arithmetic expression)*

   The set of *oriented* picture frames, which we shall call S, can be enumerated by a sequence of 6 choices: two choices among 2 alternatives for the decorations, and four choices from among 3 alternatives for the moulding colors, so S contains $2^2 3^4 = 324$ elements. However, to eliminate equivalent frames from our enumeration we will need to use Burnside's lemma. These particular frames are subject to four possible transformations: the identity e, the 180° rotation r, the horizontal flip h, and the vertical flip v. Let us assume our paintings have landscape orientation (if you assume portrait orientation, the analysis is the same, with the roles of h and v reversed).

   Every frame is invariant under the identity, so $|\operatorname{Inv}(e)| = |S| = 324$.

   Under the rotation, every single element is mapped onto an antipodal element, so in order to be invariant under r, a frame must have identical decorations, identical left-and-right mouldings, and identical top-and-bottom mouldings. Thus such an invariant can be chosen in $2\times 3\times 3 = 18$ ways, so $|\operatorname{Inv}(r)| = 18$.

   Under the horizontal flip, the decorations and the upper and lower mouldings are left in pace, while the left and right mouldings are swapped. Thus, invariants under h are all of those where the left and right mouldings are the same, which can be done in $2^2 3^3 = 108$ ways, so $|\operatorname{Inv}(h)| = 108$.

   Finally, the vertical swap leaves the left and right mouldings in place, while the top and bottom mouldings and decorations are flipped, so invariants under v have identical mouldings and decorations on top and bottom, which can be done in $2\cdot 3\cdot 3^2 = 54$ ways, so $|\operatorname{Inv}(v)| = 54$.

   Thus, there are

$$\frac{324+18+108+54}{4} = 126$$

   possible different picture frames.

3. **(25 points)** *There is streetside parking along a particular city block, with room for several vehicles. On any given day, the street might be occupied by pickup trucks, sedans, minivans, SUVs, and supercompacts. In order to present a favorable impression of the community's fuel-efficiency, the city council has mandated that at least one supercompact, and at most one pickup truck, shall be parked on the block. Different orderings of the vehicles on the block are considered to be different parking configurations.*

   (a) **(12 points)** *Letting a_n represent the number of ways to park n cars on the block in accordance with the city council's mandate, find a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.*
   The selections for sedans, minivans, and SUVs are free, so each of these is characterized by the generating function $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = e^x$. The selection of supercompacts has no constant term, since zero subcompacts are forbidden, yielding $x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = e^x$. The selection of pickup trucks is finite generating function $1 + x$. Putting all of these together, we see that the generating function for the sequence of number of distinct parking configurations is:

$$g(x) = (1+x)(e^x)^3(e^x - 1) = e^{4x} - e^{3x} + xe^{4x} - xe^{3x}$$

   (b) **(13 points)** *Either using your generating function or by other means, determine how many different possible ways there are to park 5 cars in the block. (Note: you may leave your result as an unsimplified arithmetic expression)*
   Solving this via generating functions, we may attempt to determine the coefficient of x^5 in the generating function determined above:

$$g(x) = e^{4x} - e^{3x} + xe^{4x} - xe^{3x} = \sum_{n=0}^{\infty} \frac{(4x)^n}{n!} - \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(4x)^n x}{n!} - \sum_{n=0}^{\infty} \frac{(3x)^n x}{n!}$$

   which can be simplified as such:

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} 4^n \frac{x^n}{n!} - \sum_{n=0}^{\infty} 3^n \frac{x^n}{n!} + \sum_{n=0}^{\infty} 4^n \frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} 3^n \frac{x^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} (4^n - 3^n) \frac{x^n}{n!} + \sum_{n=0}^{\infty} (4^n - 3^n)(n+1) \frac{x^{n+1}}{(n+1)!} \\ &= \sum_{n=0}^{\infty} (4^n - 3^n) \frac{x^n}{n!} + \sum_{n=1}^{\infty} (4^{n-1} - 3^{n-1}) n \frac{x^n}{n!} \\ &= \sum_{n=1}^{\infty} (4^n - 3^n + n4^{n-1} - n3^{n-1}) \frac{x^n}{n!} \end{aligned}$$

   so that there are $4^n - 3^n + n4^{n-1} - n3^{n-1}$ ways to park n cars, and specifically $4^5 - 3^5 + 5 \cdot 4^4 - 5 \cdot 3^4 = 1656$ ways to park 5 cars.
   Alternatively, one could solve this with a direct counting methodology. We might divide configurations into two types: those with a pickup truck, and those without. In the first case we might consider the 4^5 free placements of our 4 types of vehicles, but then exclude the 3^5 placements which do not include a supercompact. For the second case, we might

freely choose one of the five spaces to contain a pickup, and then multiply by the dfference between the 4^4 free-placement fillings of the remaining four spaces and the 3^4 placements which we have to exclude because they don't contain a supercompact. The total is then $4^5 - 3^5 + 5(4^4 - 3^4)$ as above.

4. **(25 points)** *Find the solution to the recurrence relation* $a_n = -3a_{n-1} + 4a_{n-2} + 25$ *with initial conditions* $a_0 = 2$ *and* $a_1 = 4$.

   The associated homogeneous recurrence is $b_n = -3b_{n-1} + 4b_{n-2}$, which has characteristic polynomial $\lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1)$, so the solution to the associated homogeneous equation is $b_n = (-4)^n A + 1^n B = A(-4)^n + B$.

   A naïve choice for a particular solution to the inhomogeneous equation would be $a_n^p = C$, but this overlaps with the constant homogeneous term, so we give it a "bump" to get $a_n^p = Cn$. Then, plugging into the recurrence:

$$\begin{aligned} Cn &= -3C(n-1) + 4C(n-2) + 25 \\ Cn &= -3Cn + 3C + 4Cn - 8C + 25 \\ Cn &= Cn - 5C + 25 \\ 5C &= 25 \\ C &= 5 \end{aligned}$$

   so $a_n^p = 5n$, and as a result $a_n = A(-4)^n + B + 5n$. Plugging in values for $a_0 = 2$ and $a_1 = 4$, we get:

$$\begin{cases} 2 = a_0 = \quad A + B \\ 4 = a_1 = -4A + B + 5 \end{cases}$$

   This can be solved in a number of different ways (subtracting the two equations eliminates B, for instance), yielding $A = \frac{3}{5}$ and $B = \frac{7}{5}$ and the final formula $a_n = \frac{3(-4)^n+7}{5} + 5n$.

5. **(25 points)** *Consider the following algorithm performed on a pair of numbers x and y.*

   Algorithm FOOBAR(x, y):

   (1) If $y = 0$, output 1.

   (2) If y is odd, then let $q = \text{FOOBAR}(x, \frac{y-1}{2})$, and then output the product $q \cdot q \cdot x$.

   (3) If y is even, then let $q = \text{FOOBAR}(x, \frac{y}{2})$, and then output the product $q \cdot q$.

   (a) *Walk through the algorithm's procedure when performed on the inputs $(2, 6)$, determining its eventual output. What does this algorithm seem to do?*

   While evaluating FOOBAR$(2, 6)$, we meet the conditions for step 3 and must then assign q to be FOOBAR$(2, 3)$, so then we must evaluate FOOBAR$(2, 3)$. In evaluating this, we meet the conditions of step 2 and must then let q be FOOBAR$(2, 1)$, so we then must evaluate FOOBAR$(2, 1)$ in order to perform this task , which itself requires another invocation of step 2 to get FOOBAR$(2, 0)$, which will be 1. Working our way out of this innermost level, we see that FOOBAR$(2, 1) = 1 \cdot 1 \cdot 2 = 2$, and then that FOOBAR$(2, 3) = 2 \cdot 2 \cdot 2 = 8$, and finally that FOOBAR$(2, 6) = 8 \cdot 8 = 64$, so our final output is 64.

   This procedure uses a bisection method to calculate exponents. Fundamentally, it uses the principle that for even n, $x^n = (x^{\frac{n}{2}})^2$, while for odd n, $x^n = (x^{\frac{n-1}{2}})^2 x$.

(b) *Parameterized in terms of y (instead of the customary n), what is the runtime of this algorithm, in big-O notation?*

Running this algorithm for a specific value of y requires performing a constant number of simpe steps as well as running the same algorithm for $\lfloor \frac{y}{2} \rfloor$. After $\log_2 y$ such recursions, the parameter will be winnowed down to 0 whereupon the algorithm becomes trivial. Thus, the algorithm requires a number of steps which is a constant multiple of $\log_2 y$, so it runs in $O(\log y)$ time.