For full credit show all of your work (legibly!), unless otherwise specified. Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, factorials, permutations, and combinations.

1. **(10 points)** Let a_n be the number of ways to write n as a sum of positive integers in which each integer appears *at most twice*. For example, $a_4 = 4$, because there are the four partitions 4, $3 + 1$, $2 + 2$, and $2 + 1 + 1$ (but *not* $1 + 1 + 1 + 1$). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.

2. **(15 points)** Your company makes reversible picture frames, all the same size. These frames are rectangular but *not* square, and each frame has either a curlicue or no decoration on each of the two longest sides, as well as having the segments of moulding (the four timbers on the outside of the frame) each painted either red, white, or blue. One such frame (in a landscape orientation) might have a curlicue on bottom but not on top, while the bottom and top mouldings are blue, and the left and right mouldings are red and white respectively. Two frames are identical if one can be rotated *or reflected* onto the other, and you produce every possible frame (including boring ones, like an all-white frame without decorations). How many different frames are there? (Note: you may leave your result as an unsimplified arithmetic expression)

3. **(25 points)** There is streetside parking along a particular city block, with room for several vehicles. On any given day, the street might be occupied by pickup trucks, sedans, minivans, SUVs, and supercompacts. In order to present a favorable impression of the community's fuel-efficiency, the city council has mandated that at least one supercompact, and at most one pickup truck, shall be parked on the block. Different orderings of the vehicles on the block are considered to be different parking configurations.

   (a) **(12 points)** Letting a_n represent the number of ways to park n cars on the block in accordance with the city council's mandate, find a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

   (b) **(13 points)** Either using your generating function or by other means, determine how many different possible ways there are to park 5 cars in the block. (Note: you may leave your result as an unsimplified arithmetic expression)

4. **(25 points)** Find the solution to the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2} + 25$ with initial conditions $a_0 = 2$ and $a_1 = 4$.

5. **(25 points)** Consider the following algorithm performed on a pair of numbers x and y.

Algorithm FOOBAR(x, y):

(1) If $y = 0$, output 1.

(2) If y is odd, then let $q = $ FOOBAR$(x, \frac{y-1}{2})$, and then output the product $q \cdot q \cdot x$.

(3) If y is even, then let $q = $ FOOBAR$(x, \frac{y}{2})$, and then output the product $q \cdot q$.

(a) Walk through the algorithm's procedure when performed on the inputs $(2, 6)$, determining its eventual output. What does this algorithm seem to do?

(b) Parameterized in terms of y (instead of the customary n), what is the runtime of this algorithm, in big-O notation?