1. **(21 points)** *Answer the following questions about recurrence relations.*

   (a) **(9 points)** *Find the solution to the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with initial conditions $a_0 = 7$ and $a_1 = 12$.*

   This equation is a linear homogeneous recurrence of order 2; it can be solved using any of the standard techniques but the most straightforward is making use of the characteristic polynomial, which is $\lambda^2 = 8\lambda - 16$, which can be rearranged to $(\lambda - 4)^2 = 0$, so λ can only equal 4, with multiplicity 2 and thus the general formula for a_n is $A4^n + Bn4^n$. Then we will use the initial conditions to determine these coefficients:

$$\begin{cases} 7 = a_0 = A + 0B \\ 12 = a_1 = 4A + 4B \end{cases}$$

   The first equation gives us $A = 7$ easily; plugging it into the second, we get $4B = 12 - 28 = -16$, so $B = -4$. Thus, $a_n = 7 \cdot 4^n - 4n \cdot 4^n$.

   (b) **(12 points)** *Find the solution to the recurrence relation $r_n = 3r_{n-1} - 2r_{n-2} + 5n$ with initial conditions $r_0 = 5$ and $r_1 = 6$.*

   We note that the associated homogeneous equation is $b_n = 3b_{n-1} - 2b_{n-2}$, which has associated characteristic polynomial $\lambda^2 - 3\lambda + 2$ which has roots 1 and 2, so the solution to this homogeneous recurrence is $A + B2^n$.For a particular solution to the inhomogeneous equation, we note that the inhomogeneous part is $5n$. Naïvely we might adopt the template $C + Dn$, but this overlaps the homogeneous term A and must be bumped up to $Cn + Dn^2$. Plugging this particular solution into the recurrence gives

$$\begin{aligned} Cn + Dn^2 &= 3[C(n-1) + D(n-1)^2] - 2[C(n-2) + D(n-2)^2] + 5n \\ Cn + Dn^2 &= 3[(D - C) + (C - 2D)n + Dn^2] - 2[(4D - 2C) + (C - 4D)n + Dn^2] + 5n \\ Cn + Dn^2 &= (C - 5D) + (C + 2D + 5)n + Dn^2 \end{aligned}$$

$$(5D - C) - 2Dn = 5n$$

   So clearly $D = -\frac{5}{2}$ and then $C = 5D = -\frac{25}{2}$, giving a particular inhomogeneous solution of $-\frac{25}{2}n - \frac{5}{2}n^2$, and thus a general solution of $r_n = A + B2^n - \frac{25}{2}n - \frac{5}{2}n^2$. We shall solve for A and B using the initial conditions:

$$\begin{cases} 5 = r_0 = A + B \\ 6 = r_1 = A + 2B - 15 \end{cases}$$

   Subtracting the first equation from the second gives $B = 16$, and then plugging back into the first yields $A = -11$, for a solution of $r_n = 16 \cdot 2^n - 11 - \frac{25}{2}n - \frac{5}{2}n^2$.

2. **(8 points)** *Let a_n represent the number of ways of writing n as a sum of positive integers in which each positive integer appears exactly 0, 2, or 3 times. For instance, $a_8 = 3$ because 8 can be written according to those rules as $4+4$, $3+3+1+1$, or $2+2+2+1+1$. Find a formula for the generating function $\sum_{n=0}^{\infty} a_n x^n$.*

   The selection function for inclusion of 1s is $1 + x^2 + x^3$; the selection function for inclusion of 2s is $1 + x^4 + x^6$, and so forth, so the generating function for the partition process as a whole is

$$\prod_{i=0}^{\infty}(1 + x^{2i} + x^{3i}).$$

3. **(15 points)** *I have four red, two green, and one white ping-pong ball I wish to put into a long, narrow tube.*

   (a) **(4 points)** *How many different ways could the balls be ordered within the tube?*

   There are seven locations for the white ball; the two green balls can them be placed in two of the remaining six locations in $\binom{6}{2}$ ways; then the red balls are forced into a specific position so there are $7\binom{6}{2} = 105$ possible placements; note this quantity could also be identified as the multinomial coefficient $\binom{7}{4,2,1}$.

   (b) **(6 points)** *How many different ways are there to order the balls within the tube if I insist that neither all the red balls be clumped together, nor that both of the green balls be together?*

   We must excludse those placements where the green balls are together; considering the green balls as a single unit, there are $\binom{6}{4,1,1} = 30$ such configurations. We must also exclude those where the red balls form a single unit, of which there are $\binom{4}{1,2,1} = 12$. Finally, we must re-include the over-removed overlap of these two families, where both the red and green balls form a unit; this occurs in $\binom{3}{1,1,1} = 6$ ways, so the total number of configurations is

$$\binom{7}{4,2,1} - \binom{6}{4,1,1} - \binom{4}{1,2,1} + \binom{3}{1,1,1} = 69.$$

   (c) **(5 points)** *How many ways are there to place the balls in the tube if I consider an ordering within the tube to be identical to its reversal?*

   As seen in part (a), there are 105 *oriented* arrangements. Now let us consider, preparatory to using Burnside's Lemma on a two-element group, which of those arrangements are invariant over the reversal (a flip, or a 180° rotation, as the case may be): the white ball would need to occupy the middle position, and the red and green would be arranged symmetrically about it, with two green and one red on each side; it is easy to see that there are exactly three invariant arrangements under the flip, so by Burnside's lemma, there are $\frac{105+3}{2} = 54$ arrangements up to reversal-equivalency.

4. **(16 points)** *An individual-sized crudité plate is considered to be attractively balanced if it has between 3 and 5 carrots inclusive, fewer than 7 pieces of celery, at least one piece of broccoli, any number of red-pepper slivers, and at least 4 snap-pea pods.*

   (a) **(8 points)** *Let a_n represent the number of possible attractively balanced plates with n vegetables. Find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.*

   Selection functions for the individual types of vegetables are easily built: carrots are associated with $x^3+x^4+x^5$, celery with $1+x+\cdots+x^6$, broccoli with $x+x^2+x^3+\cdots = \frac{x}{1-x}$, red-pepper with $1+x+x^2+\cdots = \frac{1}{1-x}$, and snap-peas with $x^4+x^5+x^6+\cdots = \frac{x^4}{1-x}$. Their product is going to be the moderately messy expression:

$$\frac{(x^3+x^4+x^5)(1+x+x^2+x^3+x^4+x^5+x^6)x^5}{(1-x)^3}$$

   Particularly for the purposes of the next question, it is almost certainly worthwhile to rephrase those two finite geometric series more concisely: $x^3+x^4+x^5 = \frac{x^3-x^6}{1-x}$, and $1+x+x^2+\cdots+x^6 = \frac{1-x^7}{1-x}$, giving the somewhat more tractable form

$$\frac{(x^3-x^6)(1-x^7)x^5}{(1-x)^5} = \frac{x^8-x^{11}-x^{15}+x^{18}}{(1-x)^5}.$$

(b) **(8 points)** *Determine the number of different attractively balanced plates which can possibly be made with 18 vegetables.*

We might use the generating function to do this, or you could revert to more straightforward inclusion-exclusion methods. Using the generating function, we note that

$$\frac{x^8 - x^{11} - x^{15} + x^{18}}{(1-x)^5} = (x^8 - x^{11} - x^{15} + x^{18}) \sum_{n=0}^{\infty} \binom{n+4}{4} x^n$$

from which we wish to extract the coefficient of x^{18}. There are four terms which contribute: $x^8 \cdot \binom{14}{4} x^{10}$, $-x^{11} \cdot \binom{11}{4} x^7$, $-x^{15} \cdot \binom{7}{4} x^3$, and $x^{18} \cdot \binom{4}{4} x^0$.Thus the coefficient of x^{18} in this generating function is $\binom{14}{4} - \binom{11}{-}\binom{7}{4} + \binom{4}{4} = 637$.

Alternatively, we could use inclusion-exclusion if we dislike generating functions. Let us pre-emptively set aside 3 carrots, one broccoli, and four snap-peas, leaving 10 as-yet-undetermined veggies. There are $\binom{10+4}{4}$ ways to assign those vegetables without restrictions, but we have restrictions on the number of carrots and celery sticks. We specifically forbid having three (additional to our pre-emptively selected three) carrots and seven celery stalks. We can calculate the number of assignments violating our carrot provision by pre-emptively making three of our ten veggies carrots, leaving 7 unassigned and free to be assigned in $\binom{7+4}{4}$ ways. Likewise we can calculate the number violating our celery provision by assigning 7 celery stalks, leaving 3 unassigned which could be assigned in $\binom{3+4}{4}$ ways. Finally, we must consider the possibility that we have both 7 celery stalks and 3 carrots among these 10 vegetables, which is a circumstance violating both conditions and thus removed twice; we have to add this count back in so as not to over-exclude. Thus, our total is $\binom{10+4}{4} - \binom{7+4}{4} - \binom{3+4}{4} + 1$, as above.

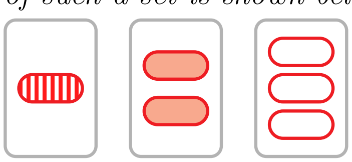
It is also *possible* to solve this problem with either the carrot or celery or both finite geometric series unsimplified, or, equivalently, to address the restriction with a casewise argument rather than an inclusion-exclusion argument, but doing so is extremely messy.

5. **(25 points)** *A SET® deck contains cards with four different attributes: number, color, symbol, and fill. Each attribute has three possibilities: for instance, cards can be red, green, or purple, depict one, two, or three shapes, that shape could be a diamond, oval, or squiggle, and the shape could be hollow, striped, or filled (some example cards are shown in the questions below, although colors are printed in greyscale).*

   (a) **(3 points)** *How many different cards are there?*

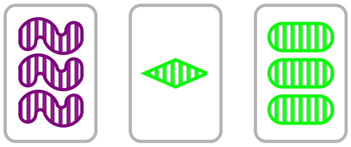
   There are four properties, and three possible choices within each property, giving $3^4 = 81$ cars in all.

   (b) **(6 points)** *One type of "set" in the game is an unordered collection of three cards which are the same color and symbol, but with all different numbers and shadings. An example of such a set is shown below. How many different sets of this type are there?*



   The set is unordered, but the three different numbers which must be present enforce a "distinguishability" among the cards. Then we select a common number (in one of 3 ways), a common symbol (in one of 3 ways), and an association of shadings with numbers (in $3! = 6$ ways), so there are a total of $3 \cdot 3 \cdot 6 = 54$ such sets.

(c) **(8 points)** *How many possible ways are there to build an* ordered *collection of three distinct cards such that exactly two have the same number, all three have the same shading, and there are no restrictions on color or shape (an example is below)?*



It is easier to actually find an *unordered* collection, and then multiply by the six possible orderings. We choose any of three shadings to have in common, any of three numbers for the doubleton, any of the two remaining numbers for the singleton, and then we must assign colors and shapes. There are 9 possible color-shape pairs, and we assign one of them to the singleton, and two of them to the doubleton (which we can do in $\binom{9}{2}$ ways. Thus, there are

$$3 \cdot 3 \cdot 2 \cdot 9\binom{9}{2} = 5832$$

unordered collections, which are associated with $5832 \cdot 6 = 34992$ possible ordered collections.

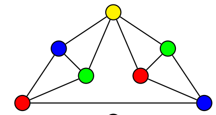
(d) **(8 points)** *How many unordered sets of* five *distinct cards have at least one card in each shape and shading (with no requirements on number or color)?*

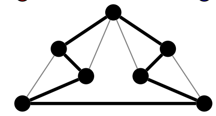
We shall use inclusion-exclusion to establish the appropriate forbiddences; let U be our universe of $\binom{81}{5}$ completely free selections of five cards without order, and let A_1, A_2, and A_3 be those selections which fail to have a squiggle, oval, or diamond, while B_1, B_2, and B_3 are sets of selections without one of the three shadings. We thus want to find out how many elements of U lie in *none* of these sets. Fortunately, all the A_i and B_i have similar analyses: since a single symbol or shading is excluded in each, there is a pool of 54 cards each could draw from, so $|A_i| = |B_i| = \binom{54}{5}$. For pairwise intersections, note that $|A_i \cap A_j| = |B_i \cap B_j| = \binom{27}{5}$, since constraint to a single symbol or a single shading leaves only 27 possible cards. On the other hand, $|A_i \cap B_j| = \binom{36}{5}$, since elements of this set use two shades, two symbols, three colors, and three numbers, for a total of 36 possibilities.

Finally, we also need to look at $A_i \cap A_j \cap B_k$; here we have a pool of 18 cards so $|A_i \cap A_j \cap B_k| = |A_i \cap B_j \cap B_k| = \binom{18}{5}$, and lastly $|A_i \cap A_j \cap B_k \cap B_\ell| = \binom{9}{5}$. Thus, putting it all togehter, our desired quantity will be

$$\binom{81}{5} - 6\binom{54}{5} + 6\binom{27}{5} + 9\binom{36}{5} - 18\binom{18}{5} + 9\binom{9}{5} = 10369620$$

6. **(16 points)** *Let G be the graph illustrated to the right. Answer the following questions. You may label the original graph, if desired.*





(a) **(6 points)** *Prove that $\chi(G) = 4$.*

We must show both that four colors suffice and that three colors are necessary. The latter is pretty easy to show because this graph contains several 3-cliques (a.k.a. triangles), whose vertices must all be different colors; we have two triangles joined end-to-end at the bottom, and it is easy to see that, if we try to use only three colors, we will inevitably end up having all three adjacent to the top vertex. To show the latter, we might explicitly present a coloring using four colors, which is depicted on the graph shown.

(b) **(4 points)** *Demonstrate that this graph is Hamiltonian.*

G will be Hamiltonian; an example of a Hamiltonian circuit is depicted to the left.

(c) **(6 points)** *Is this graph Eulerian? Why or why not?*

G will not be Eulerian because several vertices (in fact, almost all of them!) have odd degree.

7. **(14 points)** *Let a_n represent the number of ways to fly n red, white, and blue pennants on a flagpole (where order of the pennants matters) such that there is at least one red pennant, an odd number of white pennants, and no more than 2 blue pennants.*

(a) **(8 points)** *Find a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.*

The selection function for red pennants is $x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = e^x - 1$. The selection for white pennants is $x + \frac{x^3}{6} + \frac{x^5}{120} = \sinh x = \frac{e^x - e^{-x}}{2}$; the selection for blue pennants is the finite power series $1 + x + \frac{x^2}{2}$. Assembling all these together, we get the exponential generating function

$$(e^x - 1)\frac{e^x - e^{-x}}{2}\left(1 + x + \frac{x^2}{2}\right) =$$
$$\frac{2e^{2x} - 2 - 2e^x - 2e^{-x} + 2xe^{2x} - 2x - 2xe^x - 2xe^{-x} + x^2e^{2x} - x^2 - x^2e^x - x^2e^{-x}}{4}$$

(b) **(6 points)** *How many ways are there to fly 6 pennants?*

Three of the 12 terms in the generating function above are constants which don't contirubte to the degree-six term; let us look at what coefficient each of the remaining 9 terms above contributes towards the $\frac{x^6}{6!}$ term: looking term by term, we get:

$$\frac{1}{4}\left(\frac{2(2x)^6}{6!} - \frac{2x^6}{6!} - \frac{2(-x)^6}{6!} + \frac{2x(2x)^5}{5!} - \frac{2xx^5}{5!} - \frac{2x(-x)^5}{5!} + \frac{x^2(2x)^4}{4!} - \frac{x^2x^4}{4!} - \frac{x^2(-x)^4}{4!}\right)$$

which we can rearrange factoring out the $\frac{x^6}{6!}$ term, introducing new factors in the numerator and denominator where only 5! or 4! is present:

$$\frac{1}{4}\left(2^7 - 2 - 2 + 6 \cdot 2^6 - 6 \cdot 2 + 6 \cdot 2 + 6 \cdot 5 \cdot 2^4 - 6 \cdot 5 - 6 \cdot 5\right)\frac{x^6}{6!}$$

yielding a total of 232 possible arrangements.

8. **(10 points)** *Consider the following algorithm performed on a permutation π of the numbers $1, \ldots, n$; we use $\pi[i]$ to denote the ith term of the permutation.*

   (1) Let $x = 0$ and $i = 1$.

   (2) If $i = n$, stop and output x.

   (3) Let $j = i + 1$.

   (4) If $j > n$, go to step 8.

   (5) If $\pi[j] < \pi[i]$, increase the value of x by 1.

   (6) Increase the value of j by 1.

   (7) Return to step 4.

   (8) Increase the value of i by 1.

   (9) Return to step 2.

   (a) **(4 points)** *Walk through the algorithm's procedure when performed on $\pi = (3, 5, 1, 4, 2)$. What does this algorithm seem to do?*

   We can build a table showing the values, over time, of i, j, and x; for brevity we will only include those steps where a value changes.

Step	i	j	x
(1)	1		0
(3)		2	
(6)		3	
(5)			1
(6)		4	
(6)		5	
(5)			2
(6)		6	
(8)	2		
(3)		3	
(5)			3
(6)		4	
(5)			4
(6)		5	
(5)			5
(6)		6	
(8)	3		
(6)		4	
(6)		5	
(6)		6	
(8)	4		
(6)		5	
(5)			6
(6)		6	
(8)	5		
(2)	OUTPUT: 6		

This procedure counts the number of pairs in the permutation where a larger number precedes a smaller one (these are called "inversions" in enumerative combinatorics of permutations). It outputs 6 in this case, for instance, because 3 is before both 2 and 1, 5 is before 1, 4, and 2, and 4 is before 2.

(b) **(6 points)** *Give a big-O estimate of the number of operations, in terms of the length n of the permutation π, which this algorithm takes to perform its task.*

Since there are two nested loops (the value of i ranging from 1 to n and the value of j ranging from $i+1$ to $n+1$), we'd venture that this is quadratic, a.k.a. $O(n^2)$ time. This is the expected runtime for an algorithm which involves comparing every single pair in a list on length n, since there are $\binom{n}{2}$ pairs to compare, and $\binom{n}{2} = O(n^2)$.

It is worth noting, although it is outside the scope of the question asked, that counting inversions in a permutation need not be a quadratic-time procedure. There is a divide-and-conquer algorithm which counts inversions in each half of the permutation and then counts the numbers of inversions between the two halves; that implementation would take $O(n \log n)$ time.

9. **(16 point bonus, 8 each)** *On the back of this sheet, prove either (or both!) of the following statements combinatorially:*

- *For any positive integer n,* $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1}$.

The left side of this equation counts the number of ways to select an *even* number of elements out of the set $\{1, 2, \ldots, n\}$, while the right side counts the number of ways to select an *odd* number of elements. We can show that these two quantities are equal by putting their associated sets into a one-to-one correspondence. If $\mathcal{A}$ is the collection of even-cardinality subsets of $\{1, \ldots, n\}$ and $\mathcal{B}$ is the collection of odd-cardinality subsets of $\{1, \ldots, n\}$, we can build the simple bijection from $\mathcal{A}$ to $\mathcal{B}$:

$$f(S) = \begin{cases} S - \{1\} \text{ if } 1 \in S \\ S \cup \{1\} \text{ if } 1 \notin S \end{cases}$$

i.e., the function f toggles the inclusion of 1 in the set S, including it if absent and excluding it if present. Clearly this function is a bijection, since it's in fact its own inverse, and furthermore it is easy to show it maps $\mathcal{A}$ to $\mathcal{B}$, since $|f(S)|$ will necessarily be $|S| \pm 1$, and so if $|S|$ is even, then $|f(S)|$ is odd.

- *For any positive integer n and $0 \leq k \leq \frac{n}{2}$,* $\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1}$.

The right side of this equation is clearly counting the $2k+1$-element subsets of $\{1, 2, 3, \ldots, n+1\}$; what we hope to do is show that the left side counts the same thing.

Since $2k+1$ is odd, the subsets in question have a middle element. We can thus build them in the following manner: select a value for their middle element, then select k elements below the middle, and k elements above the middle. Clearly the middle element must be at least $k+1$, and can be no more than $(n+1)-k$. Let us denote this element as $m+1$, so that m could be anything from k to $n-k$¡ and then we want to be selecting k values from $1, 2, \ldots, m$ and k values from $m+2, m+3, \ldots, n+1$. Note that there are $\binom{m}{k}$ and $\binom{n-m}{k}$ ways to complete these two processes respectively, so the number of ways to perform the whole process (selection of m, selection of k values less than m, and selection of k values greater than m) is exactly $\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k}$.