For full credit show all of your work (legibly!), unless otherwise specified. This exam is closed-notes and calculators may not be used. **Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponents, factorials, binomial coefficients, and multinomial coefficients.**

1. **(21 points)** Answer the following questions about recurrence relations.

   (a) **(9 points)** Find the solution to the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ with initial conditions $a_0 = 7$ and $a_1 = 12$.

   (b) **(12 points)** Find the solution to the recurrence relation $r_n = 3r_{n-1} - 2r_{n-2} + 5n$ with initial conditions $r_0 = 5$ and $r_1 = 6$.

2. **(8 points)** Let a_n represent the number of ways of writing n as a sum of positive integers in which each positive integer appears exactly 0, 2, or 3 times. For instance, $a_8 = 3$ because 8 can be written according to those rules as $4 + 4$, $3 + 3 + 1 + 1$, or $2 + 2 + 2 + 1 + 1$. Find a formula for the generating function $\sum_{n=0}^{\infty} a_n x^n$.

3. **(15 points)** I have four red, two green, and one white ping-pong ball I wish to put into a long, narrow tube.

   (a) **(4 points)** How many different ways could the balls be ordered within the tube?

   (b) **(6 points)** How many different ways are there to order the balls within the tube if I insist that neither all the red balls be clumped together, nor that both of the green balls be together?

   (c) **(5 points)** How many ways are there to place the balls in the tube if I consider an ordering within the tube to be identical to its reversal?

4. **(16 points)** An individual-sized crudité plate is considered to be attractively balanced if it has between 3 and 5 carrots inclusive, fewer than 7 pieces of celery, at least one piece of broccoli, any number of red-pepper slivers, and at least 4 snap-pea pods.

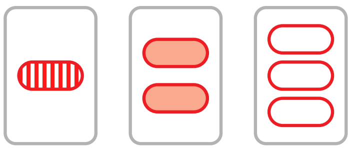
   (a) **(8 points)** Let a_n represent the number of possible attractively balanced plates with n vegetables. Find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$.

   (b) **(8 points)** Determine the number of different attractively balanced plates which can possibly be made with 18 vegetables.

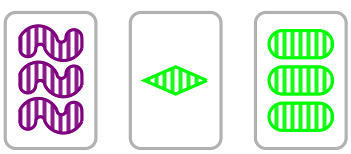
5. **(25 points)** A SET® deck contains cards with four different attributes: number, color, symbol, and fill. Each attribute has three possibilities: for instance, cards can be red, green, or purple, depict one, two, or three shapes, that shape could be a diamond, oval, or squiggle, and the shape could be hollow, striped, or filled (some example cards are shown in the questions below, although colors are printed in greyscale).

   (a) **(3 points)** How many different cards are there?

   (b) **(6 points)** One type of "set" in the game is an unordered collection of three cards which are the same color and symbol, but with all different numbers and shadings. An example of such a set is shown below. How many different sets of this type are there?



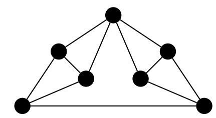
   (c) **(8 points)** How many possible ways are there to build an *ordered* collection of three distinct cards such that exactly two have the same number, all three have the same shading, and there are no restrictions on color or shape (an example is below)?



   (d) **(8 points)** How many unordered sets of *five* distinct cards have at least one card in each shape and shading (with no requirements on number or color)?

6. **(16 points)** Let G be the graph illustrated to the right. Answer the following questions. You may label the original graph, if desired.

   (a) **(6 points)** Prove that $\chi(G) = 4$.



   (b) **(4 points)** Demonstrate that this graph is Hamiltonian.

   (c) **(6 points)** Is this graph Eulerian? Why or why not?

7. **(14 points)** Let a_n represent the number of ways to fly n red, white, and blue pennants on a flagpole (where order of the pennants matters) such that there is at least one red pennant, an odd number of white pennants, and no more than 2 blue pennants.

   (a) **(8 points)** Find a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

   (b) **(6 points)** How many ways are there to fly 6 pennants?

8. **(10 points)** Consider the following algorithm performed on a permutation π of the numbers $1, \ldots, n$; we use $\pi[i]$ to denote the ith term of the permutation.

   (1) Let $x = 0$ and $i = 1$.

   (2) If $i = n$, stop and output x.

   (3) Let $j = i + 1$.

   (4) If $j > n$, go to step 8.

   (5) If $\pi[j] < \pi[i]$, increase the value of x by 1.

   (6) Increase the value of j by 1.

   (7) Return to step 4.

   (8) Increase the value of i by 1.

   (9) Return to step 2.

   (a) **(4 points)** Walk through the algorithm's procedure when performed on $\pi = (3, 5, 1, 4, 2)$. What does this algorithm seem to do?

   (b) **(6 points)** Give a big-O estimate of the number of operations, in terms of the length n of the permutation π, which this algorithm takes to perform its task.

9. **(16 point bonus, 8 each)** On the back of this sheet, prove either (or both!) of the following statements combinatorially:

   - For any positive integer n, $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1}$.
   - For any positive integer n and $0 \le k \le \frac{n}{2}$, $\sum_{m=k}^{n-k} \binom{m}{k}\binom{n-m}{k} = \binom{n+1}{2k+1}$.