1. **(10 points)** *You find that you need to buy 22 hats. The hat shop has 4 different varieties of hat: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.*

   (a) **(5 points)** *One example of a hat purchase would be: 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes. How many different possible ways are there for you to purchase 22 hats?*

   We may consider an "order" as a placement of some number of unlabeled balls in each of four distinct boxes representing different types of hats (e.g. we could communicate an order of 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes by putting 10 balls in a "stetson" box, 3 in a "beret" box, and 9 in the "stovepipe" box, while the "pillbox" box remains empty). Thus, each purchase can be associated with a distribution of 22 balls among 4 boxes. The count for doing so is a standard enumeration statistic, which we know to be $\binom{22+4-1}{4-1} = \binom{25}{3} = 2300$; alternatively, we could consider the number of ways to place 3 dividers among 22 hats, so that the hats are partitioned into 4 (possibly empty) groups, which will be declared to represent different styles of hat. This would be enumerated with $\binom{22+3}{3}$, as above.

   (b) **(5 points)** *Suppose you want to select your lot of 22 hats so that there are at least 3 hats of each type. How many ways are there to fulfill these instructions?*

   This situation is as in the first part of this problem, except that we constrain each box to contain at least three balls, and do this by pre-emptively assigning 12 balls; this would result in the statistic $\binom{25-12}{3} = \binom{13}{3} = 286$.

2. **(9 points)** *Prove by induction that* $1 + 2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 1$ *for every integer* $n \geq 1$.

For the base step, let us note that $1 + 2 = 3 = 2^2 - 1$, so the above statement is clearly true for $n = 1$ (in fact, it is also true for $n = 0$, and that could be used as the base step with equal validity). Now, given the inductive hypothesis $1 + 2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 1$, we wish to show that $1 + 2 + 4 + 8 + 16 + \cdots + 2^{n+1} = 2^{n+2} - 1$. We start with the inductive hypothesis, and perform algebra until we demonstrate our desired consequence:

$$\begin{aligned}
1 + 2 + 4 + 8 + 16 + \cdots + 2^n &= 2^{n+1} - 1 \\
1 + 2 + 4 + 8 + 16 + \cdots + 2^n + 2^{n+1} &= 2^{n+1} - 1 + 2^{n+1} \\
1 + 2 + 4 + 8 + 16 + \cdots + 2^n + 2^{n+1} &= 2 \cdot 2^{n+1} - 1 \\
1 + 2 + 4 + 8 + 16 + \cdots + 2^n + 2^{n+1} &= 2^{n+2} - 1
\end{aligned}$$

3. **(10 points)** *Show that for* $0 < k < n$, $\sum_{i=k}^{n} \binom{i}{k}$ *and* $\binom{n+1}{k+1}$ *count the same objects and are thus equal.*

The binomial coefficient $\binom{n+1}{k+1}$ clearly counts the $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$. We shall show that the sum $\sum_{i=k}^{n} \binom{i}{k}$ counts the same thing.

Given a particular $(k+1)$-element subset of $\{1, 2, \ldots, n+1\}$, we might be able to characterize it by the value of its largest element, which could plausibly be anything from $k+1$ to $n+1$; let's consider how many such sets there are with a particular largest element $i+1$. Then the remaining k elements of the subset we are looking at would be drawn from $\{1, 2, \ldots, i\}$, which can be done in $\binom{i}{k}$ ways. Summing over the entire range of possible values of i, we get the total of $\sum_{i=k}^{n} \binom{i}{k}$ possible $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$.

4. **(25 points)** *You are asked to assign your six subordinates (Alice, Bob, Carla, Dave, Ed, and Fiona) to 3 specific projects (codenamed Runcible, Screaming Fist, and Valis).*

   (a) **(5 points)** *How many ways are there to do this if you can assign people freely?*

   An assignation of the subordinates can be broken into 6 subtasks consisting of the assignment of each individual subordinate. We could assign Alice to any of 3 projects, Bob to any of 3, Carla to each of 3, and so forth, for a total of $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^6 = 729$ possible assignments.

   (b) **(10 points)** *How many ways are there to do this if you can assign people freely, as long as Carla and Fiona are not assigned to the same project?*

   There are two effective approaches to this problem (and possibly more). We could take the result from part (a), and subtract out those projects in which Carla and Fiona are working on the same project. We can count these by assigning each of Alice, Bob, Dave, and Ed to a project in one of 3 ways each, and then assigning the Carla-Fiona team to a project in one of 3 ways, for a total of 3^5 assignments in which Carla and Fiona are on the same team; subtracting these invalid assignments from the 3^6 total assignments gives $3^6 - 3^5 = 486$.

   Alternatively, you could assign the first five subordinates freely, and then for Fiona, be required to choose a project Carla is *not* on: regardless of which project Carla is on, Fiona will be left with 2 options, so we have $3^5 \cdot 2 = 486$ possible assignments.

   (c) **(10 points)** *How many ways are there to do this if each project must receive at least one worker (but there is now no restriction on placing Carla and Fiona on the same job)?*

   This is a canonical example of surjective mapping, which, as we have seen in class, is well-described with an inclusion-exclusion technique. We might let X be the set of all assignments, and then exclude therefrom the sets A in which the Runcible project is empty, B in which the Screaming Fist project is empty, and C in which Valis is empty. We saw in part (a) that $|X| = 3^6$. If one project is empty, the other two are the only viable choices for each assignment, so $|A| = |B| = |C| = 2^6$. Likewise, if any two projects are empty, the remaining choice is the only one possible for an assignment, so $|A \cap B| = |A \cap C| = |B \cap C| = 1^6$. It is impossible for all three projects to be empty, so $|A \cap B \cap C| = 0$. Thus, using inclusion-exclusion, we can find the set of surjective mappings to be

$$|X - A - B - C| = 3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 - 0 = 540$$

5. **(10 points)** *For the purposes of this question, the English language contains 5 vowels and 21 consonants; also, we call a string of letters a "word" even if it is nonsensical, like the five-letter word "QREFG". How many 6-letter words are there in which exactly 3 letters are vowels?*

   We may build such a six-letter word by performing 3 subprocesses in sequence: first, we pick 3 of the 6 positions to contain vowels, in any of $\binom{6}{3}$ ways. Then, in order, we pick a specific vowel to inhabit each of these spaces, in any of 5^3 ways. Finally, we pick a consonant for each of the remaining 3 spaces, in any of 21^3 ways, for a total of $\binom{6}{3}5^3 21^3 = 23152500$ different words.

6. **(8 points)**

(a) **(4 points)** *Determine the coefficient of xyz^3 in $(6x - 3y + 2z)^5$.*
We know that the multinomial expansion yields the term $\binom{5}{3,1,1}(6x)(-3y)(2z)^3$, so segregating out the constants yields $6(-3)(2^3)\binom{5}{3,1,1}xyz^3 = -2880xyz^3$ so the coefficient is -2880.

(b) **(4 points)** *Simplify the expression*

$$\binom{n}{0} + 3\binom{n}{1} + 9\binom{n}{2} + 27\binom{n}{3} + \cdots + 3^n\binom{n}{n}.$$

We know that

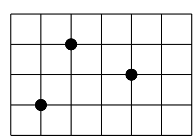
$$(1+x)^n = \binom{n}{0} + x\binom{n}{1} + x^2\binom{n}{2} + x^3\binom{n}{3} + \cdots + x^n\binom{n}{n}.$$

Substituting in $x = 3$ yields the expression asked about on the right side of the equation:

$$(1+3)^n = \binom{n}{0} + 3\binom{n}{1} + 3^2\binom{n}{2} + 3^3\binom{n}{3} + \cdots + 3^n\binom{n}{n}.$$

So this sum is 4^n.

7. **(10 points)** *How many direct paths are there from the lower left corner to the upper right corner of the following grid which* do not pass through any two of the three marked points*?*



The total number of gridwalks is enumerated by the set of lists of instructions consisting of four "up" and six "right" steps; such lists can be enumerated by $\binom{10}{4}$, since you must choose four of the ten instructions in sequence to be "up".

From this number we wish to subtract those gridwalks passing through two of the points. Note that it is impossible to pass through both $(2,3)$ and $(4,2)$, since that would require downwards steps; we thus must only concern ourself with walks through $(1,1)$ and $(2,3)$, and walks through $(1,1)$ and $(4,2)$. The former can be broken into three subwalks: a walk from $(0,0)$ to $(1,1)$ in any of $\binom{2}{1}$ ways, a walk from $(1,1)$ to $(2,3)$ in any of $\binom{3}{1}$ ways, and a walk from $(2,3)$ to $(6,4)$ in any of $\binom{5}{1}$ ways; thus this first family of invalid walks has $\binom{2}{1}\binom{3}{1}\binom{5}{1}$ members. Likewise, walks through $(1,1)$ and $(4,2)$ can be broken into the three subwalks: a walk from $(0,0)$ to $(1,1)$ in any of $\binom{2}{1}$ ways, a walk from $(1,1)$ to $(4,2)$ in any of $\binom{4}{1}$ ways, and a walk from $(4,2)$ to $(6,4)$ in any of $\binom{4}{2}$ ways; thus this family of invalid walks has $\binom{2}{1}\binom{4}{1}\binom{4}{2}$ members, so the total is:

$$\binom{10}{4} - \binom{2}{1}\binom{3}{1}\binom{5}{1} - \binom{2}{1}\binom{4}{1}\binom{4}{2} = 132$$