For full credit show all of your work (legibly!), unless otherwise specified. Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, factorials, permutations, and combinations.

1. **(10 points)** You find that you need to buy 22 hats. The hat shop has 4 different varieties of hat: stetsons, berets, stovepipes, and pillboxes. Hats within a single variety are identical.

   (a) **(5 points)** One example of a hat purchase would be: 10 stetsons, 3 berets, 9 stovepipes, and no pillboxes. How many different possible ways are there for you to purchase 22 hats?

   (b) **(5 points)** Suppose you want to select your lot of 22 hats so that there are at least 3 hats of each type. How many ways are there to fulfill these instructions?

2. **(9 points)** Prove by induction that $1 + 2 + 4 + 8 + 16 + \cdots + 2^n = 2^{n+1} - 1$ for every integer $n \geq 1$.

3. **(10 points)** Show that for $0 < k < n$, $\sum_{i=k}^{n} \binom{i}{k}$ and $\binom{n+1}{k+1}$ count the same objects and are thus equal.

4. **(25 points)** You are asked to assign your six subordinates (Alice, Bob, Carla, Dave, Ed, and Fiona) to 3 specific projects (codenamed Runcible, Screaming Fist, and Valis).

   (a) **(5 points)** How many ways are there to do this if you can assign people freely?

   (b) **(10 points)** How many ways are there to do this if you can assign people freely, as long as Carla and Fiona are not assigned to the same project?

   (c) **(10 points)** How many ways are there to do this if each project must receive at least one worker (but there is now no restriction on placing Carla and Fiona on the same job)?

5. **(10 points)** For the purposes of this question, the English language contains 5 vowels and 21 consonants; also, we call a string of letters a "word" even if it is nonsensical, like the five-letter word "QREFG". How many 6-letter words are there in which exactly 3 letters are vowels?

6. **(8 points)**

   (a) **(4 points)** Determine the coefficient of xyz^3 in $(6x - 3y + 2z)^5$.

   (b) **(4 points)** Simplify the expression

$$\binom{n}{0} + 3\binom{n}{1} + 9\binom{n}{2} + 27\binom{n}{3} + \cdots + 3^n\binom{n}{n}$$

7. **(10 points)** How many direct paths are there from the lower left corner to the upper right corner of the following grid which *do not pass through any two of the three marked points*?