1. **(25 points)** *The Wichway Catering Company provides five different types of sandwiches to your organization, but has some peculiar rules about how many sandwiches of each type can be in an order. They will provide any number of turkey sandwiches and pastrami sandwiches, but insist that every order must contain at least 10 vegetarian sandwiches and no more than 8 roast beef. Finally, an order can only have between 5 and 15 ham sandwiches. So, for instance, an order might consist of no turkey, 15 pastrami, 12 vegetarian, 1 roast beef, and 8 ham (which would be 36 sandwiches in total).*

   (a) **(10 points)** *Letting a_n represent the number of different possible ways to order n sandwiches, find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   The selections for turkey and pastrami are free, so each of these is characterized by the generating function $1 + x + x^2 = x^3 + \cdots = \frac{1}{1-x}$. The selection of vegetarian sandwiches starts with the term of exponent 10, yielding $x^{10} + x^{11} + x^{12} + \cdots = \frac{x^{10}}{1-x}$. The selection of roast beef is the finite generating function $1 + x + x^2 + \cdots + x^8 = \frac{1-x^9}{1-x}$, and likewise the selection of ham is $x^5 + x^6 + x^7 + \cdots + x^{15} = \frac{x^5 - x^{16}}{1-x}$. Putting all of these together, we see that the generating function for the sequence of ways to get sandwiches of these 5 types collectively is:

$$\frac{x^{10}(1-x^9)(x^5-x^{16})}{(1-x)^5} = \frac{x^{15} - x^{24} - x^{26} + x^{35}}{(1-x)^5}$$

   (b) **(5 points)** *What is the lowest-degree non-zero term in the power series of the generating function you determined above? What is the significance of this term?*

   The power series for this generating function would in fact have zero terms up to x^{15}; this signifies that the restrictions on this particular process are such that ordering fewer than 15 sandwiches is impossible, and the term x^{15} would have coefficient 1 because there is in fact only one way to make the minimum order (10 vegetarian and 5 ham).

   (c) **(10 points)** *Either using your generating function or by other means, determine how many different possible ways there are to place an order for 100 sandwiches.*

   Solving this via generating functions, we may attempt to determine the coefficient of z^{100} in the generating function determined above:

$$\frac{z^{15} - z^{24} - z^{26} + z^{35}}{(1-z)^5} = (z^{15} - z^{24} - z^{26} + z^{35}) \sum_{n=0}^{\infty} \binom{n+4}{4} z^n$$

   And in this summation, we can determine which products of the fixed powers of z in the first factor and the powers of z in the power series have total exponent 10). We may take, for instance, $z^{15} \cdot \binom{85+4}{4} z^{85} = \binom{89}{4} z^{100}$, and likewise we will get associated with the other terms in the first factor the products $-z^{24} \cdot \binom{76+4}{4} z^{76} = -\binom{80}{4} z^{100}$, $-z^{26} \cdot \binom{74+4}{4} z^{74} = -\binom{78}{4} z^{100}$, and $z^{35} \cdot \binom{65+4}{4} z^{65} = -\binom{69}{4} z^{100}$, for a total z^{100} coefficient of $\binom{89}{4} - \binom{80}{4} - \binom{78}{4} + \binom{69}{4} = 298122$.

   Alternatively, one could solve this with inclusion-exclusion. We would start by preemptively assigning 10 vegetarian sandwiches and 5 ham, leaving 85 of our 100 left to be assigned. We then let X be the set of all such assignments; with a balls-and-wall paradigm we may assert that $|X| = \binom{85+4}{4} = \binom{89}{4}$. Now, from this, we wish to exclude the set A of all distributions with more than 8 roast beef, and the set B of all distributions with more than 15 ham (i.e., more than 10 *additional* ham, as we have already allocated 5 prior to defining X). To find $|A|$, we would pre-emptively assign 9 roast beef sandwiches, forcing violation of the roast-beef condition, and assign the remaining 76 sandwiches in $\binom{76+4}{4} = \binom{80}{4}$ ways; likewise, we can find $|B|$ by pre-emptively assigning 11 more ham sandwiches (bringing the total to 16), and assign the remaining 74 sandwiches in $\binom{74+4}{4} = \binom{78}{4}$ ways. Finally, $|A \cap B|$ is found by preemptively performing both assignments, which, togethe rwith our initial assignment of 15 sandwiches, leaves only 65 sandwiches left to select, in $\binom{65+4}{4} = \binom{69}{4}$ ways. Invoking inclusion-exclusion, we find that

$$|X - A - B| = \binom{89}{4} - \binom{80}{4} - \binom{78}{4} + \binom{69}{4} = 298122.$$

2. **(15 points)** *Find the following generating functions:*

   (a) **(5 points)** *Let a_n be the number of ways to place n* distinct *objects in 4 boxes so that each box contains at least 2 items. Determine a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.*

   Each individual box can be filled, in isolation, exactly one way with 2 items, exactly one way with 3, and so forth, giving each box its own generating function $1\frac{x^2}{2!} + 1\frac{x^3}{3!} + 1\frac{x^4}{4!} + \cdots = e^x - 1 - x$. The generating function for the process as a whole is the product of four such functions, so it is $(e^x - 1 - x)^4$.

   (b) **(10 points)** *Let b_n be the number of ways to write n as a sum of (not necessarily distinct) powers of 2 (e.g. 1, 2, 4, 8, 16, etc.). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n x^n$.*

   The generating function for selection of some quantity of ones to use is $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$. Likewise, the generating function for selection of some quantity of twos is $1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$. Analagously, the selection of fours, eights, and sixteens would be represented by $\frac{1}{1-x^4}$, $\frac{1}{1-x^8}$, and $\frac{1}{1-x^{16}}$, with the pattern continuing, until we see that the final generating function is

$$\frac{1}{(1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})\cdots} = \frac{1}{\prod_{i=0}^{\infty}(1-x^{2^i})}$$

3. **(20 points)** *Find the particular solution to the recurrence relation $a_n = 4a_{n-1} + 27n$ with initial condition $a_0 = 7$.*

4. **(20 points)** *Consider the following algorithm performed on a pair of numbers a and b.*

   Algorithm Mystery(a, b):

   (1) If $a = 0$, output b.

   (2) If $b = 0$, output a.

   (3) If $a \le b$, output the result of performing Mystery$(a, b - a)$.

   (4) If $a > b$, output the result of performing Mystery$(a - b, b)$.

   (a) *Walk through the algorithm's procedure when performed on the inputs $(60, 84)$, determining its eventual output. What does this algorithm seem to do?*

While evaluating MYSTERY$(60, 84)$, we meet the conditions for step 3 and must then determine MYSTERY$(60, 24)$; in evaluating this, we meet the conditions of step 4 and must determine MYSTERY$(36, 24)$, which itself requires another invokation of step 4 to get MYSTERY$(12, 24)$. We invoke step 3 twice now, first to get MYSTERY$(12, 12)$ and then to get MYSTERY$(12, 0)$, whereupon step 2 gives us (at long last!) an output of 12.

This is an implementation of the Euclidean algorithm for determining greatest common divisors; steps (3) and (4) decrease the parameters while maintaining the GCD until eventually one parameter is zero.

(b) *Letting $n = \max(a, b)$, what is the runtime of this algorithm, in big-O notation?*

Looking at what might happen if we run the procedure on $(n, 1)$, which is the worst-case scenario, the recursion will laboriously decrease the first parameter by one n times before getting an answer; thus the runtime is linear, or $O(n)$.

5. **(15 points)** *Find the closed form of the recurrence relation given by initial conditions $b_0 = 4$, $b_1 = 9$, and $b_n = 6b_{n-1} - 9b_{n-2}$ for $n \geq 2$.*

Taking the template $b_n = \lambda^n$, the recurrence becomes $\lambda^n = 6\lambda^{n-1} - 9\lambda^{n-2}$, which can be algebraically simplified to $\lambda^2 - 6\lambda + 9 = 0$, so that the solution to this polynomial is $\lambda = 3$ with a multiplicity of 2. To get two distinct choices of b_n from this, we assert not only that $b_n = 3^n$ solves the recurrence, but also that its slightly "bumped" form $b_n = n3^n$ works. Thus, the general solution to the recurrence above is the linear combination $b_n = A3^n + Bn3^n$, and we must use the initial conditions to determine these coefficients. Plugging in $b_0 = 4$, we find that $4 = A3^0 + B03^0 = A$, so $A = 4$. Then since $b_1 = 9$, $9 = A3^1 + B13^1 = 3A + 3B$, so $B = 9 - 3A = -3$. Thus our closed form will be $b_n = 4 \cdot 3^n - 3n \cdot 3^n$.

6. **(10 points)** *You are building circular bracelets with 6 beads on them; you have beads in red, yellow, and green. You want to have at least one bead of each color on every bracelet, and two bracelets are considered to be identical if one can be produced by flipping or rotating the other. How many different bracelets are possible?*

Huh, this is basically the same question as #4 on the problem set. No answer for y'all until next week then!