For full credit show all of your work (legibly!), unless otherwise specified. Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, factorials, permutations, and combinations.

1. **(25 points)** The Wichway Catering Company provides five different types of sandwiches to your organization, but has some peculiar rules about how many sandwiches of each type can be in an order. They will provide any number of turkey sandwiches and pastrami sandwiches, but insist that every order must contain at least 10 vegetarian sandwiches and no more than 8 roast beef. Finally, an order can only have between 5 and 15 ham sandwiches. So, for instance, an order might consist of no turkey, 15 pastrami, 12 vegetarian, 1 roast beef, and 8 ham (which would be 36 sandwiches in total).

   (a) **(10 points)** Letting a_n represent the number of different possible ways to order n sandwiches, find a formula for the ordinary generating function $\sum_{n=0}^{\infty} a_n z^n$.

   (b) **(5 points)** What is the lowest-degree non-zero term in the power series of the generating function you determined above? What is the significance of this term?

   (c) **(10 points)** Either using your generating function or by other means, determine how many different possible ways there are to place an order for 100 sandwiches.

2. **(15 points)** Find the following generating functions:

   (a) **(5 points)** Let a_n be the number of ways to place n *distinct* objects in 4 boxes so that each box contains at least 2 items. Determine a formula for the exponential generating function $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

   (b) **(10 points)** Let b_n be the number of ways to write n as a sum of (not necessarily distinct) powers of 2 (e.g. 1, 2, 4, 8, 16, etc.). Determine a formula for the ordinary generating function $\sum_{n=0}^{\infty} b_n x^n$.

3. **(20 points)** Find the particular solution to the recurrence relation $a_n = 4a_{n-1} + 27n$ with initial condition $a_0 = 7$.

4. **(20 points)** Consider the following algorithm performed on a pair of numbers a and b.

   Algorithm MYSTERY(a, b):

   (1) If $a = 0$, output b.

   (2) If $b = 0$, output a.

   (3) If $a \leq b$, output the result of performing MYSTERY$(a, b - a)$.

   (4) If $a > b$, output the result of performing MYSTERY$(a - b, b)$.

   (a) Walk through the algorithm's procedure when performed on the inputs $(60, 84)$, determining its eventual output. What does this algorithm seem to do?

   (b) Letting $n = \max(a, b)$, what is the runtime of this algorithm, in big-O notation?

5. **(15 points)** Find the closed form of the recurrence relation given by initial conditions $b_0 = 4$, $b_1 = 9$, and $b_n = 6b_{n-1} - 9b_{n-2}$ for $n \geq 2$.

6. **(10 points)** You are building circular bracelets with 6 beads on them; you have beads in red, yellow, and green. You want to have at least one bead of each color on every bracelet, and two bracelets are considered to be identical if one can be produced by flipping or rotating the other. How many different bracelets are possible?