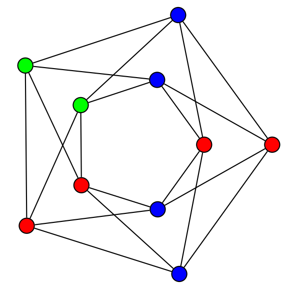
1. **(15 points)** *Let G be the graph illustrated to the right. Answer the following questions. You may label the original graph, if desired.*



   (a) **(6 points)** *Is this graph Eulerian? Why or why not?*

   G will be Eulerian because every vertex has degree 4, which is even, and because it is connected.

   (b) **(9 points)** *Demonstrate that $\chi(G) = 3$.*

   We must show both that three colors suffice and that three colors are necessary. The former is pretty easy to show because a 2-coloring would basically require us to choose a color for one vertex, then the opposite color for its neighbors, and continue alternating colors until the whole graph is colored, but very quickly we would run into trouble because of the odd-length cycles.

   To show the latter, we might explicitly present a coloring using three colors, which is depicted on the graph shown.

2. **(21 points)** *Answer the following questions about recurrence relations.*

   (a) **(9 points)** *Find the solution to the recurrence relation $a_n = a_{n-1} + 20a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 73$.*

   This equation is a linear homogeneous recurrence of order 2; it can be solved using any of the standard techniques but the most straightforward is making use of the characteristic polynomial, which is $\lambda^2 = \lambda + 20$, which can be rearranged to $(\lambda - 5)(\lambda + 4) = 0$, so λ can equal 5 or -4 and thus the general formula for a_n is $A5^n + B(-4)^n$. Then we will use the initial conditions to determine these coefficients:

$$\begin{cases} 2 = a_0 = A + B \\ 73 = a_1 = 5A - 4B \end{cases}$$

   Adding four times the first equation to the second gives $81 = 9A$, which can be solved to get $9 = A$; then since $B = 2 - A$, $B = -7$ giving a final answer of $a_n = 9 \cdot 5^n - 7(-4)^n$.

   (b) **(12 points)** *Find the particular solution to the recurrence relation $b_n = b_{n-1} + 20b_{n-2} + 28 \cdot 3^n$ with initial conditions $b_0 = -10$ and $b_1 = -32$.*

   We note that the associated homogeneous equation, which is the same as the equation in the previous part of this question, has solution $A5^n + B(-4)^n$. For a particular solution to the inhomogeneous equation, we note that the inhomogeneous part is $28 \cdot 3^n$ so we will want to adopt the template $b_n^p = A3^n$ (note that this does not overlap the homogeneous solution, so no "bumping" is required). Plugging this particular solution into the recurrence gives

$$\begin{aligned} A3^n &= A3^{n-1} + 20A3^{n-2} + 28 \cdot 3^n \\ 9A &= 3A + 20A + 28 \cdot 9 \\ -14A &= 28 \cdot 9 \\ A &= -2 \cdot 9 = -18 \end{aligned}$$

so $b_n^p = -18 \cdot 3^n$; putting in the homogeneous solution we find that the general solution to the inhomogeneous recurrence is $b_n = A5^n + B(-4)^n - 18 \cdot 3^n$. We shall solve for A and B using the initial conditions:

$$\begin{cases} -10 = b_0 = A + B - 18 \\ -32 = b_1 = 5A - 4B - 54 \end{cases}$$

And, as previously, adding four times the first euqation to the second eliminates B, giving $-72 = 9A - 126$, or $A = 6$. Then $B = 8 - A = 2$, so the final equation is $b_n = 6 \cdot 5^n + 2(-4)^n - 18 \cdot 3^n$.

3. **(25 points)** *I have a deck of cards with three different properties on each card: number, suit, and color. Numbers range from 1–5, there are 3 different suits, and there are 4 colors. This deck has one representative of each number/suit/color combination.*

   (a) **(3 points)** *How many different cards are there?*

   There are five possible numbers, three different suits, and four different colors, making $5 \cdot 3 \cdot 4 = 60$ possible cards in all.

   (b) **(6 points)** *How many unordered hands of 3 cards are possible in which no* number *appears twice?*

   We might build such a hand using the following process: select three numbers, with no intrinsic order to the selection. Then, since the numbers are all different, choose a suit and color for the lowest, then the middle, and then the highest card. The first step can be completed in $\binom{5}{3}$ ways, and each of the following three can be completed in $3 \cdot 4$ ways, for a total of $\binom{5}{3} \cdot 12^3 = 17280$. (The last arithmetic calculation is not necessary, and the expression $\binom{5}{3} \cdot 12^3$ would be an acceptable answer)

   (c) **(8 points)** *How many unordered hands of 5 cards are there with "two pairs", i.e. two different numbers which appear twice, and one other number which appears exactly once.*

   We might perform this task by selecting two "pair numbers" in $\binom{5}{2}$ ways, and then a third "singleton number" from among the 3 remaining numbers. Then to each of the pairs in turn (the lower and then the higher, to distinguish them), we would choose two of the 12 number/suit combinations to assign to the cards; finally, we would choose one of the 12 number/suit combinations for the singleton. Assembling all this together, we get that there are

   $$\binom{5}{2} \cdot 3\binom{12}{2}^2 \cdot 12 = 1568160$$

   hands. Probably the arithmetic calculation at the end is not worth doing by hand.

   (d) **(8 points)** *How many unordered hands of 5 cards are there which contain at least one card from each of the three suits?*

   We may use inclusion-exclusion to find the number of such hands. There are obviously $\binom{60}{5}$ unrestricted hands, since there are 60 cards. From this, we wish to exclude those which fail to use one suit. There are three possible suits to fail to use, and 40 cards from outside that suit, so there are $3 \cdot \binom{40}{5}$ cases to exclude, but now we must re-include their overlapis, in which only one suit is used; there are 3 suits to use and 20 cards within each suit, so there are $3 \cdot \binom{20}{5}$ cases to re-include. The total is thus

   $$\binom{60}{5} - 3\binom{40}{5} + 3\binom{20}{5} = 3534000.$$

Other, explicit casewise breakdowns are certainly possible; for instance, one could consider the hands broken down into two families: those with three cards of a single suit and one of the other two, and those with two card in each of two suits and one of the third suit. Each family has three variations depending on which suit is present in a different quantity; the first family then has $\binom{20}{3} \cdot 20^2$ possible hands in each variation, and the second has $\binom{20}{2}^2 \cdot 20$ hands, for a total of

$$3\binom{20}{3} \cdot 20^2 + 3\binom{20}{2}^2 \cdot 20 = 3534000.$$

Note that these two different approaches, although resulting in quite different arithmetic expressions, nonetheless evaluate to give the same number (as they should). Also, evaluating the arithmetic expression to determine the value is probably not worth doing by hand, in either case.

4. **(25 points)** *We are buying perennials for our garden, and decide that we want a mix of dahlias, lilies, mums, and foxglove. We decide that, for aesthetic purposes, we want no more than 3 foxglove, no fewer than 5 lilies, and at least one dahlia; we might have as many mums as suit our purposes. Let a_n be the number of ways to place an order for n flowers conforming to these conditions.*

   (a) **(8 points)** *Find a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.*

   We can express this generating function as a product of generating functions associated with the selection process for each individual variety of flower. Since we can have no more than 3 foxglove, the generating function for foxglove selection is actually the polynomial $1 + z + z^2 + z^3$; we could alternatively express this, if we wish, in the rational form $\frac{1-z^4}{1-z}$. The selection process for lilies requires at least five, so its generating function is $z^5 + z^6 + z^7 + \cdots = \frac{z^5}{1-z}$. Likewise, our dahlia generating function is $z + z^2 + z^3 + \cdots = \frac{z}{1-z}$, and our mums have the free-selection function $1 + z + z^2 + \cdots = \frac{1}{1-z}$. To get the generating function for a_n, we multiply the generating rfunctions for these individual processes, getting one of the two algebraically equivalent expressions $\frac{z^6-z^{10}}{(1-z)^4}$ or $\frac{z^6+z^7+z^8+z^9}{(1-z)^3}$.

   (b) **(4 points)** *What is the degree of the smallest nonzero term in your calculation above? What is the significance of this term?*

   The smallest nonzero term of the series is the z^6 term, i.e. a term of degree 6. This is because $a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = 0$, which signifies that our desired restrictions are impossible to meet with fewer than six flowers.

   (c) **(13 points)** *If we want our garden to have 20 flowers, how many different ways could we make our purchase? You may use the generating function from part (a) if desired.*

   We may answer this question with the generating function in either form, or with a direct calculation.

   Using $\sum_{n=0}^{\infty} a_n z^n = \frac{z^6-z^{10}}{(1-z)^4}$, we may expand the latter term into

$$(z^6 - z^{10}) \sum_{n=0}^{\infty} \binom{n+3}{3} z^n$$

   and we note that there are two z^{20} terms which arise: $z^6 \cdot \binom{17}{3} z^{14}$ and $-z^{10} \cdot \binom{13}{3} z^{10}$. Thus a_{20}, which is the coefficient of z^{20} in the generating function, is $\binom{17}{3} - \binom{13}{3} = 394$.

Alternatively, with $\sum_{n=0}^{\infty} a_n z^n = \frac{z^6+z^7+z^8+z^9}{(1-z)^3}$, we can expand to get

$$(z^6 + z^7 + z^8 + z^9) \sum_{n=0}^{\infty} \binom{n+2}{2} z^n$$

and find that four terms of the product contribute towards z^{20} when collected: $z^6 \cdot \binom{16}{2} z^{14}$, $z^7 \cdot \binom{15}{2} z^{13}$, $z^8 \cdot \binom{14}{2} z^{12}$, and $z^9 \cdot \binom{13}{2} z^{11}$. Then a_{20} will be the sum of all these coefficients, $\binom{16}{2} + \binom{15}{2} + \binom{14}{2} + \binom{13}{2} = 394$.

Either of the above expressions could also be gotten using direct invocation of a balls-and-walls paradigm. We pre-emptively assign 6 balls collectively to the lily and dahlia boxes, leaving 14 balls to be assigned with no more than three in the foxglove box. We might use an exclusion approach, where we take all $\binom{14+3}{3}$ assignations to the four boxes and subtract out those with at least four foxgloves, which using pre-emptivce assignment of these four, is $\binom{10+3}{3}$. Our difference is $\binom{17}{3} - \binom{13}{3}$ as above. Alternatively, we could divide into four cases based on how many foxgloves there are and use balls-and-walls on the other three assignations: if no foxgloves, there are $\binom{14+2}{2}$ ways to assign; if one, then there are $\binom{13+2}{2}$ ways to assign the rest; $\binom{12+2}{2}$ ways with two foxgloves, and $\binom{11+2}{2}$ with three. Then we have $\binom{16}{2} + \binom{15}{2} + \binom{14}{2} + \binom{13}{2}$ distributions in total.

5. **(24 points)** *We are placing objects in 5* distinguishable *boxes, such that the first 3 boxes must receive at least one item, which the other two may receive as many as you wish.*

   (a) **(6 points)** *If we have exactly nine identical items, how many ways are there to distribute the items?*

   We pre-emptively assign 3 items to the first three boxes, leaving 6 left to be placed. Using a balls-and-walls paradigm we have 6 balls and 4 walls for a total of $\binom{10}{6} = 210$ placements.

   (b) **(8 points)** *Find an exponential generating function $\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$, where b_n represents the number of ways to distribute n distinguishable objects among these boxes.*

   Because the first three boxes cannot be empty, each of them is associated with the exponential generating function $z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots = e^z - 1$. The two free boxes are each associated with the exponential generating function $1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = e^z$. THe process as a whole is thus characterized by the exponential generating function which is a product of these five sub-processes, which will be $(e^z - 1)^3 (e^z)^2 = e^{5z} - 3e^{4z} + 3e^{3z} - e^{2z}$.

   (c) **(6 points)** *If we have exactly nine distinguishable items, how many ways are there to distribute the items? Your answer probably should* not *be arithmetically simplified.*

   Let us note that, using the power series expansion:

$$e^{5z} - 3e^{4z} + 3e^{3z} - e^{2z} = \sum_{n=0}^{\infty} \frac{(5z)^n}{n!} - 3\frac{(4z)^n}{n!} + 3\frac{(3z)^n}{n!} - \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} (5^n - 3 \cdot 4^n + 3 \cdot 3^n - 2^n) \frac{z^n}{n!}$$

   so the coefficient b_9 of $\frac{z^9}{9!}$ will be $5^9 - 3 \cdot 4^9 + 3 \cdot 3^9 - 2^9 = 1225230$.

   (d) **(4 points)** *Explain, without explicit arithmetic computation, why your answer to part (b) must be divisible by 6. Why is your answer* not *necessarily divisible by 120?*

   Because our items are distinguishable and the first three boxes contain at least one item apiece, each placement could be permuted to five different placements by shuffling the labels on our first 3 boxes; by this permutation the placements are grouped into families of size exactly 6, and since there are an integer number of families, $\frac{b_9}{6}$ must thus be an integer.

We could *not* extend this same notion to shuffling all five boxes to get families of size 120, however. If the fourth or fifth boxes are empty, then some of the 120 permutations would not be valid (swapping an empty box into the first, second, or third position), and if *both* the fourth and fifth boxes are empty, then some of the 120 permutations would not lead to a different placement (swapping the fourth and fifth boxes with each other). So, grouping the placements by permutability on all five boxes would not create families of identical size: some would have size 6, some size 24, and some size 120.

6. **(15 points)** *Let us consider anagrams of the word BONOBO; note that we are counting all arrangements of these letters, not simply those that are English words.*

   (a) **(4 points)** *How many different anagrams does this word have?*

   There are six possible locations for the N, and three of the remaining five locations are then selected for the Os; the Bs are forced into the remaining two locations, and this process can be completed in $6 \cdot \binom{5}{3} = 60$ ways. Other orderings, or the multinomial coefficient $\binom{6}{3,2,1}$ will produce the same numerical answer.

   (b) **(6 points)** *How many anagrams are there which do not have the two "B"s together?*

   If we were to consider the two Bs as a single unit, then we have $\binom{5}{3,1,1} = 20$ placements, which subtracted from the total above gives $60 - 20 = 40$ placements *without* the Bs together.

   (c) **(5 points)** *How many anagrams have neither the two "B"s together, nor all three of the "O"s together?*

   As above $\binom{5}{3,1,1} = 20$ placements have the two Bs together, and if we collect all three Os as a single unit, there are $\binom{4}{1,2,1} = 12$ placements. However, excluding both of these will double-count those which have both the Bs and Os together, and we must reinclude the $\binom{3}{1,1,1} = 6$ such placements to get $60 - 20 - 12 + 6 = 34$ total anagrams.

7. **(10 points)** *Consider the following algorithm performed on a number n.*

   (1) Let $c = 0$, and let $q = 1$.

   (2) If $n = 0$, output c.

   (3) If n is odd, then assign $c + q$ to c and decrement n by 1.

   (4) Assign $\frac{n}{2}$ to n.

   (5) Assign $10 \cdot q$ to q.

   (6) Return to step 2.

   (a) **(4 points)** *Walk through the algorithm's procedure when performed on the number 140. What does this algorithm seem to do?*

   We can build a table showing the values, over time, of n, c, and q; note the step numbers will repeat, since we return to step 2 regularly, and often steps 2 and 3 will have no effect and will be omitted.

Step	n	c	q
(1)	140	0	1
(4)	70		
(5)			10
(4)	35		
(5)			100
(3)	34	100	
(4)	17		
(5)			1000
(3)	16	1100	
(4)	8		
(5)			10000
(4)	4		
(5)			100000
(4)	2		
(5)			1000000
(4)	1		
(5)			10000000
(3)	0	10001100	
(4)	0		
(5)			100000000
(2)	OUTPUT: 1001100		

This procedure outputs a representation (encoded in decimal for printing) of n converted to binary notation: 140 in base 10 is 1001100 in binary.

(b) **(6 points)** *Give a big-O estimate of the number of operations, in terms of n, which this algorithm takes to perform its task.*

Since over the course of a constant number of steps, n is halved, and this sequence is repeated until n dwindles to zero, we know that the procedure is logarithmic, with runtime $O(\log n)$.

8. **(8 point bonus)** *On the back of this sheet, prove combinatorially that for any positive integer i, it is the case that $\sum_{i=1}^{n} i^2\binom{n}{i} = n2^{n-1} + n(n-1)2^{n-2}$.*

The left side can be combinatorially thought of as selecting a subset S of $\{1, 2, 3, \ldots, n\}$ with at least one element (but with any number $i \geq 1$ of elements possible), and then selecting (possibly identical) elements x and y of S; what we are building is the triple (S, x, y) with $x, y \in S \subseteq \{1, \ldots, n\}$. In less abstract terms, we might consider this process to be the selection of a committee (of any size) from among n people, and then selecting a chairperson and a secretary for the committee (who might be the same person).

On the right side we perform the same procedure in a different order, and with a casewise separation. There are two plausible cases we shall consider: either $x = y$ (i.e. the chair and secretary are the same), or $x \neq y$ (the roles are distinct). In the first case, we select x (and simultaneously y) from among the n possibilities, and then build $S - \{x\}$ completely freely from the remaining $n - 1$ numbers, which can be done in 2^{n-1} ways since we may choose to include or exclude each number. In the second case, we select x from among the n possibilities, y from among the remaining $n - 1$ choices, and finally choose to include or exclude each of the remaining $n - 2$ numbers in $S - \{x, y\}$ in any of 2^{n-2} ways. This process, we see, can be completed in $n2^{n-1} + n(n-1)2^{n-2}$ ways.

Since the two processes have the same set of results, the number of ways they can be completed must be the same, and so $\sum_{i=1}^{n} i^2 \binom{n}{i} = n2^{n-1} + n(n-1)2^{n-2}$.