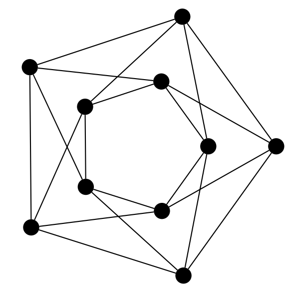
For full credit show all of your work (legibly!), unless otherwise specified. This exam is closed-notes and calculators may not be used. **Answers need not be completely reduced unless otherwise stated, and may be left in terms of sums, differences, products, quotients, exponents, factorials, binomial coefficients, and multinomial coefficients.**

1. **(15 points)** Let G be the graph illustrated to the right. Answer the following questions. You may label the original graph, if desired.



   (a) **(6 points)** Is this graph Eulerian? Why or why not?

   (b) **(9 points)** Demonstrate that $\chi(G) = 3$.

2. **(21 points)** Answer the following questions about recurrence relations.

   (a) **(9 points)** Find the solution to the recurrence relation $a_n = a_{n-1} + 20a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 73$.

   (b) **(12 points)** Find the particular solution to the recurrence relation $b_n = b_{n-1} + 20b_{n-2} + 28 \cdot 3^n$ with initial conditions $b_0 = -10$ and $b_1 = -32$.

1	/ 15
2	/ 21
3	/ 25
4	/ 25
5	/ 24
6	/ 15
7	/ 10
8	/ (8)
Σ	/135

3. **(25 points)** I have a deck of cards with three different properties on each card: number, suit, and color. Numbers range from 1–5, there are 3 different suits, and there are 4 colors. This deck has one representative of each number/suit/color combination.

   (a) **(3 points)** How many different cards are there?

   (b) **(6 points)** How many unordered hands of 3 cards are possible in which no *number* appears twice?

   (c) **(8 points)** How many unordered hands of 5 cards are there with "two pairs", i.e. two different numbers which appear twice, and one other number which appears exactly once.

   (d) **(8 points)** How many unordered hands of 5 cards are there which contain at least one card from each of the three suits?

4. **(25 points)** We are buying perennials for our garden, and decide that we want a mix of dahlias, lilies, mums, and foxglove. We decide that, for aesthetic purposes, we want no more than 3 foxglove, no fewer than 5 lilies, and at least one dahlia; we might have as many mums as suit our purposes. Let a_n be the number of ways to place an order for n flowers conforming to these conditions.

   (a) **(8 points)** Find a formula for the generating function $\sum_{n=0}^{\infty} a_n z^n$.

   (b) **(4 points)** What is the degree of the smallest nonzero term in your calculation above? What is the significance of this term?

   (c) **(13 points)** If we want our garden to have 20 flowers, how many different ways could we make our purchase? You may use the generating function from part (a) if desired.

5. **(24 points)** We are placing objects in 5 *distinguishable* boxes, such that the first 3 boxes must receive at least one item, which the other two may receive as many as you wish.

   (a) **(6 points)** If we have exactly nine identical items, how many ways are there to distribute the items?

   (b) **(8 points)** Find an exponential generating function $\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}$, where b_n represents the number of ways to distribute n distinguishable objects among these boxes.

   (c) **(6 points)** If we have exactly nine distinguishable items, how many ways are there to distribute the items? Your answer probably should *not* be arithmetically simplified.

   (d) **(4 points)** Explain, without explicit arithmetic computation, why your answer to part (b) must be divisible by 6. Why is your answer *not* necessarily divisible by 120?

6. **(15 points)** Let us consider anagrams of the word BONOBO; note that we are counting all arrangements of these letters, not simply those that are English words.

   (a) **(4 points)** How many different anagrams does this word have?

   (b) **(6 points)** How many anagrams are there which do not have the two "B"s together?

   (c) **(5 points)** How many anagrams have neither the two "B"s together, nor all three of the "O"s together?

7. **(10 points)** Consider the following algorithm performed on a number n.

   (1) Let $c = 0$, and let $q = 1$.

   (2) If $n = 0$, output c.

   (3) If n is odd, then assign $c + q$ to c and decrement n by 1.

   (4) Assign $\frac{n}{2}$ to n.

   (5) Assign $10 \cdot q$ to q.

   (6) Return to step 2.

   (a) **(4 points)** Walk through the algorithm's procedure when performed on the number 140. What does this algorithm seem to do?

   (b) **(6 points)** Give a big-O estimate of the number of operations, in terms of n, which this algorithm takes to perform its task.

8. **(8 point bonus)** On the back of this sheet, prove combinatorially that for any positive integer i, it is the case that $\sum_{i=1}^{n} i^2 \binom{n}{i} = n2^{n-1} + n(n-1)2^{n-2}$