1. **(8 points)** *Estimate the following values using appropriate linear approximations.*

   (a) **(4 points)** $\sqrt{25.07}$.

   We observe that this number is very close to 25, which has a nice square root. We thus use a linear approximation to the function $f(x) = \sqrt{x}$ near the point $a = 25$. Since $f'(x) = \frac{1}{2\sqrt{x}}$, we ascertain that $f(25) = 5$ and $f'(25) = \frac{1}{10}$ and so

$$f(25 + 0.07) \approx f(25) + 0.07f'(25) = 5 + \frac{0.07}{10} = 5.007$$

   Note that the actual value of $\sqrt{25.07}$ is approximately 5.0069951. so 10.001 is accurate to five decimal places.

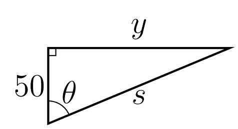
   (b) **(4 points)** $(0.993)^5$.

   We observe that this number is very close to 1, which has an easily calculated fifth power. We thus use a linear approximation to the function $f(x) = x^5$ near the point $a = 1$. Since $f'(x) = 5x^4$, we ascertain that $f(1) = 1$ and $f'(1) = 5$ and so

$$f(1 - 0.007) \approx f(1) - 0.007f'(1) = 1 - 0.035 = 0.965$$

   Note that the actual value of 0.993^5 is approximately 0.965486582. so 0.965 is a good but not great approximation.

2. **(15 points)** *A prison guard using a spotlight has located an escaped prisoner 50 meters north and 120 meters east of their current position. The prisoner is running eastwards at 3 meters per second.*



   (a) **(9 points)** *How quickly is the prisoner moving away from the guard?*

   The above image depicts the relationship in question; the guard's north-south distance from the prisoner is a constant 50 meters, while the east-west displacement y and straight-line distance s are changing. From the given information, we know that $\frac{dy}{dt} = 3$, and we wish to know $\frac{ds}{dt}$. Since there is a right triangle with legs 50 and y and hypotenuse s, we also know that $50^2 + y^2 = s^2$. Differentiating both sides of this relationship with respect to time:

$$\frac{d}{dt}\left(50^2 + y^2\right) = \frac{d}{dt}s^2$$
$$\frac{dy}{dt}\frac{d}{dy}\left(50^2 + y^2\right) = \frac{ds}{dt}\frac{d}{ds}s^2$$
$$\frac{y\frac{dy}{dt}}{s} = \frac{ds}{dt}$$

   and we know that y and $\frac{dy}{dt}$ are 120 and 3 at the moment; a little computation shows that $s = \sqrt{50^2 + 120^2} = 130$, and so $\frac{ds}{dt} = \frac{120\cdot 3}{130} = \frac{36}{13}$ meters per second.

(b) **(6 points)** *How quickly should the guard be turning the spotlight to keep it trained on the fugitive?*

We have the same scenario and labels as above, but now we seek to find $\frac{d\theta}{dt}$. There are several trigonometric relationships we can use to describe θ, but since y and 50 are the quantities in the triangle about which we know the most, the relationship $\tan\theta = \frac{y}{50}$ is particularly fruitful. Differentiating it with respect to time:

$$\begin{aligned}
\frac{d}{dt}\tan\theta &= \frac{d}{dt}\frac{y}{50} \\
\frac{d\theta}{dt}\frac{d}{d\theta}\tan\theta &= \frac{\frac{dy}{dt}}{50} \\
\frac{d\theta}{dt}\sec^2\theta &= \frac{\frac{dy}{dt}}{50} \\
\frac{d\theta}{dt} = \frac{\frac{dy}{dt}\cos^2\theta}{50} &= \frac{\frac{dy}{dt}\left(\frac{50}{s}\right)^2}{50} = \frac{3\frac{2500}{16900}}{50} = \frac{150}{16900}
\end{aligned}$$

Note that this result is about 0.009 in the unusual units of radians-per-second; if you want a more palatable unit, this is about 30 degrees per minute.

3. **(12 points)** *Find* $\frac{d}{dx}\frac{e^{\arctan x}}{\csc x}$.

This is a quotient on its outermost level; looking ahead, we might note that the numerator includes the composition $e^{\arctan x}$. We thus reasonably expect to use both the quotient rule and the chain rule in calculating this derivative.

$$\begin{aligned}
\frac{d}{dx}\frac{e^{\arctan x}}{\csc x} &= \frac{(\csc x)\frac{d}{dx}(e^{\arctan x}) - e^{\arctan x}\frac{d}{dx}(\csc x)}{\csc^2 x} \\
&= \frac{(\csc x)\frac{d}{dx}e^u - e^{\arctan x}(-\csc x\cot x)}{\csc^2 x} \text{ where } u = \arctan x \\
&= \frac{\csc x\frac{du}{dx}\frac{d}{du}e^u + e^{\arctan x}\csc x\cot x}{\csc^2 x} \\
&= \frac{\csc x\left(\frac{1}{1+x^2}\right)e^u + e^{\arctan x}\csc x\cot x}{\csc^2 x} \\
&= \frac{\frac{\csc x e^{\arctan x}}{1+x^2} + e^{\arctan x}\csc x\cot x}{\csc^2 x} \\
&= e^{\arctan x}\left(\frac{\sin x}{1+x^2} + \cos x\right)
\end{aligned}$$

The last line is a simplification and is not necessary.

4. **(14 points)** *The* cissoid of Diocles *is a curve satisfying the equation* $x(x^2+y^2) = 4y^2$.

(a) **(10 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*

We implicitly differentiate both sides of the equation, and process it until only the terms

x, y, and $\frac{dy}{dx}$ remain:

$$\begin{aligned}
\frac{d}{dx}\left(x(x^2+y^2)\right) &= \frac{d}{dx}\left(4y^2\right) \\
(x^2+y^2) + x\frac{d}{dx}(x^2+y^2) &= \frac{dy}{dx}\frac{d}{dy}\left(4y^2\right) \\
(x^2+y^2) + 2x^2 + x\frac{d}{dx}y^2 &= \frac{dy}{dx}8y \\
3x^2+y^2+x\frac{dy}{dx}\frac{d}{dy}y^2 &= \frac{dy}{dx}8y \\
3x^2+y^2+2xy\frac{dy}{dx} &= 8y\frac{dy}{dx}
\end{aligned}$$

And now we need to algebraically isolate $\frac{dy}{dx}$:

$$\begin{aligned}
3x^2+y^2+2xy\frac{dy}{dx} &= 8y\frac{dy}{dx} \\
2xy\frac{dy}{dx} - 8y\frac{dy}{dx} &= -3x^2-y^2 \\
(2xy-8y)\frac{dy}{dx} &= -3x^2-y^2 \\
\frac{dy}{dx} &= \frac{-3x^2-y^2}{2xy-8y} = \frac{3x^2+y^2}{8y-2xy}
\end{aligned}$$

(b) **(4 points)** *Find the equation of the tangent line to the curve at* $(2,-2)$*.*
The slope will be the value of $\frac{dy}{dx}$ at this specific point, which is

$$\left.\frac{dy}{dx}\right|_{(2,-2)} = \frac{3\cdot 2^2+(-2)^2}{8(-2)-2\cdot 2(-2)} = \frac{16}{-8} = -2$$

So we want a line of slope -2 through $(2,-2)$, which will have equation $y+2=-2(x-2)$.

5. **(10 points)** *Find an equation of the tangent line to the curve* $y=\frac{x^2+4}{x^2+2x}$ *at* $(2,1)$*.*

Note that, using the quotient rule, $\frac{dy}{dx} = \frac{(x^2+2x)(2x)-(x^2+4)(2x+2)}{(x^2+2x)^2}$. Evaluated at $x=2$, this will give us a slope of $\frac{32-48}{64} = \frac{-1}{4}$, so the line will have equation $y-1=\frac{-1}{4}(x-2)$.

6. **(12 points)** *Calculate* $\frac{d}{dx}\frac{(x^3+2x)(e^x+\sin x)}{\ln x+7}$.

This expression is a quotient on its outermost level; we may observe that one of the terms in the quotient is a product, so we will expect to use first the quotient rule, and then the product rule.

$$\frac{d}{dx}\frac{(x^3+2x)(e^x+\sin x)}{\ln x+7} = \frac{(\ln x+7)\frac{d}{dx}\left[(x^3+2x)(e^x+\sin x)\right]-(x^3+2x)(e^x+\sin x)\frac{d}{dx}(\ln x+7)}{(\ln x+7)^2}$$

$$= \frac{(\ln x+7)\left[\left(\frac{d}{dx}(x^3+2x)\right)(e^x+\sin x)+(x^3+2x)\frac{d}{dx}(e^x+\sin x)\right]-(x^3+2x)(e^x+\sin x)\frac{1}{x}}{(\ln x+7)^2}$$

$$= \frac{(\ln x+7)\left[(3x^2+2)(e^x+\sin x)+(x^3+2x)(e^x+\cos x)\right]-\frac{(x^3+2x)(e^x+\sin x)}{x}}{(\ln x+7)^2}$$

7. **(8 points)** *Find the maximum and minumum values of the function $f(x)=x^3-3x^2-7$ on the interval $[-2,1]$.*

$f'(x)=3x^2-6x$, which exists everywhere, so our only critical points are where $3x^2-6x=0$, which occurs when $x=0$ or $x=2$. Our candidates for maximum are thus -2, 1, 0, and 2; we can reject 2 out of hand because it is not in the interval $[-2,1]$. Of the remaining three, $f(-2)=-27$, $f(0)=-7$, and $f(1)=-9$, so 0 is a maximum and -2 a minimum.

8. **(21 points)** *Answer the following derivative-related questions.*

   (a) **(7 points)** *If $f(q)=(\operatorname{arcsec} q)(\cos q+8\ln q)$, find $f'(q)$.*

   This is an application of the product rule:

$$f'(q)=\left(\frac{d}{dq}\operatorname{arcsec} q\right)(\cos q+8\ln q)+(\operatorname{arcsec} q)\frac{d}{dq}(\cos q+8\ln q)=\frac{\cos q+8\ln q}{q\sqrt{q^2-1}}+(\operatorname{arcsec} q)(\frac{8}{q}-\sin q)$$

   (b) **(7 points)** *Find $\frac{d}{dt}\cot(e^{\arcsin t})$.*

   This is a double composition here; we may let $u=e^{\arcsin t}$ and $v=\arcsin t$, so that

$$\begin{aligned}\frac{d}{dt}\cot u &= \frac{dv}{dt}\frac{du}{dv}\frac{d}{du}\cot u\\ &= \left(\frac{d}{dt}\arcsin t\right)\left(\frac{d}{dv}e^v\right)\left(\frac{d}{du}\cot u\right)\\ &= \frac{1}{\sqrt{1-t^2}}e^v(-\csc^2 u)\\ &= \frac{-e^{\arcsin t}\csc^2 e^{\arcsin t}}{\sqrt{1-t^2}}\end{aligned}$$

   (c) **(7 points)** *Compute $\frac{d}{ds}\left(s^5-\frac{\sqrt{s}}{e^{7s}}\right)$.*

   This involves an application of the quotient rule:

$$\frac{d}{ds}\left(s^5-\frac{\sqrt{s}}{e^{7s}}\right)=5s^4-\frac{e^{7s}\frac{d}{ds}\sqrt{s}-\sqrt{s}\frac{d}{ds}e^{7s}}{(e^{7s})^2}=5s^4-\frac{\frac{e^{7s}}{2\sqrt{s}}-7\sqrt{s}e^{7s}}{e^{14s}}$$

   This can, but need not, be simplified to $5s^4+\frac{14s-1}{e^{7s}\sqrt{s}}$.

9. **(6 point bonus)** Currently Yvette is 10 miles north of the Library of Babel, walking south at 3mph, while Zachary is 1 mile east of the Library, walking east at 5mph. How soon will it be the case that the distance between them is (if only momentarily) unchanging? Do your work on the back of this sheet.