1. **(14 points)** *The* conchoid of de Sluze *is a curve satisfying the equation* $(x-1)(x^2+y^2) = 4x^2$.

   (a) **(10 points)** *Find a formula for* $\frac{dy}{dx}$ *on this curve.*
   We implicitly differentiate both sides of the equation, and process it until only the terms x, y, and $\frac{dy}{dx}$ remain:

$$\begin{aligned}
\frac{d}{dx}\left[(x-1)(x^2+y^2)\right] &= \frac{d}{dx}(4x^2) \\
\left[\frac{d}{dx}(x-1)\right](x^2+y^2) + (x-1)\frac{d}{dx}(x^2+y^2) &= 8x \\
1(x^2+y^2) + (x-1)(2x + \frac{d}{dx}y^2) &= 8x \\
(x^2+y^2) + (x-1)(2x + \frac{dy}{dx}\frac{d}{dy}y^2) &= 8x \\
(x^2+y^2) + (x-1)(2x + \frac{dy}{dx}2y) &= 8x
\end{aligned}$$

   And now we need to algebraically isolate $\frac{dy}{dx}$:

$$\begin{aligned}
x^2+y^2+(x-1)(2x+2y\frac{dy}{dx}) &= 8x \\
x^2+y^2+2x^2-2x+2xy\frac{dy}{dx}-2y\frac{dy}{dx} &= 8x \\
2xy\frac{dy}{dx}-2y\frac{dy}{dx} &= 8x-x^2-y^2-2x^2+2x \\
(2xy-2y)\frac{dy}{dx} &= 8x-x^2-y^2-2x^2+2x \\
\frac{dy}{dx} &= \frac{8x-3x^2-y^2+2x}{2xy-2y}
\end{aligned}$$

   (b) **(4 points)** *Find the equation of the tangent line to the curve at* $(3, -3)$.
   The slope will be the value of $\frac{dy}{dx}$ at this specific point, which is

$$\left.\frac{dy}{dx}\right|_{(3,-3)} = \frac{8\cdot 3 - 3\cdot 3^2 - (-3)^2 + 2\cdot 3}{2\cdot 3(-3) - 2(-3)} = \frac{-1}{2}$$

   So we want a line of slope $\frac{-1}{2}$ through $(3, -3)$, which will have equation $y+3 = \frac{-1}{2}(x-3)$.

2. **(12 points)** *Find* $\frac{d}{dx}\frac{\arcsin x}{(\sin e^x)-7}$.
   This is a quotient on its outermost level; looking ahead, we might note that the denominator includes the composition $\sin e^x$. We thus reasonably expect to use both the quotient rule and

the chain rule in calculating this derivative.

$$\begin{aligned}\frac{d}{dx}\frac{\arcsin x}{(\sin e^x)-7} &= \frac{(\sin e^x - 7)\frac{d}{dx}(\arcsin x) - \arcsin x\frac{d}{dx}(\sin e^x - 7)}{(\sin e^x - 7)^2} \\ &= \frac{(\sin e^x - 7)\frac{1}{\sqrt{1-x^2}} - \arcsin x\frac{d}{dx}\sin u}{(\sin e^x - 7)^2} \text{ where } u = e^x \\ &= \frac{\frac{(\sin e^x)-7}{\sqrt{1-x^2}} - \arcsin x\frac{du}{dx}\frac{d}{du}\sin u}{(\sin e^x - 7)^2} \\ &= \frac{\frac{(\sin e^x)-7}{\sqrt{1-x^2}} - (\arcsin x)e^x\cos u}{(\sin e^x - 7)^2} \\ &= \frac{\frac{(\sin e^x)-7}{\sqrt{1-x^2}} - (\arcsin x)e^x\cos e^x}{(\sin e^x - 7)^2}\end{aligned}$$

3. **(21 points)** *Answer the following derivative-related questions.*

   (a) **(7 points)** *If $y = e^{\sqrt{\tan\theta}}$, find $\frac{dy}{d\theta}$.*
   This is a double composition here; we may let $u = \sqrt{\tan\theta}$ and $v = \tan\theta$, so that

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{d\theta} \\ &= \left(\frac{d}{du}e^u\right)\left(\frac{d}{dv}\sqrt{v}\right)\left(\frac{d}{d\theta}\tan\theta\right) \\ &= e^u\cdot\frac{1}{2\sqrt{v}}\cdot\sec^2\theta \\ &= \frac{e^{\sqrt{\tan\theta}}\sec^2\theta}{2\sqrt{\tan\theta}}\end{aligned}$$

   (b) **(7 points)** *Compute $\frac{d}{dt}\frac{t^3-\csc t}{\arctan t}$.*
   This is an application of the quotient rule:

$$\frac{d}{dt}\frac{t^3-\csc t}{\arctan t} = \frac{\arctan t\frac{d}{dt}(t^3-\csc t) - (t^3\csc t)\frac{d}{dt}\arctan t}{\arctan^2 t} = \frac{\arctan t(3t^2+\cot t\csc t) - \frac{t^3\csc t}{1+t^2}}{\arctan^2 t}$$

   (c) **(7 points)** *If $f(x) = e^{4x}\ln x$, find $f'(x)$.*
   This is an application of the product rule:

$$f'(x) = \frac{d}{dx}\left(e^{4x}\ln x\right) = \left(\frac{d}{dx}e^{4x}\right)\ln x + e^{4x}\frac{d}{dx}\ln x = 4e^{4x}\ln x + \frac{e^{4x}}{x}$$

4. **(10 points)** *Find an equation of the tangent line to the curve $y = \frac{x^2-3\ln x}{x-2}$ at $(1,-1)$.*

Note that, using the quotient rule, $\frac{dy}{dx} = \frac{(x-2)(2x-\frac{3}{x})-(x^2-3\ln x)}{(x-2)^2}$. Evaluated at $x = 1$, this will give us a slope of $\frac{(1-2)(2\cdot 1-\frac{3}{1})-(1^2-3\ln 1)}{(1-2)^2} = 1$, so the line will have equation $y+1 = 1(x-1)$.

5. **(8 points)** *Find the absolute maximum and minumum values of the function* $f(x) = 5 + 54x - 2x^3$ *on the interval* $[0, 4]$.

   $f'(x) = 54 - 6x^2$, which exists everywhere, so our only critical points are where $54 - 6x^2 = 0$, which occurs when $x^2 = 9$ or $x = \pm 3$. Our candidates for maximum are thus -3, 3, 0, and 4; we can reject -3 out of hand because it is not in the interval $[0, 4]$. Of the remaining three, $f(3) = 113$, $f(0) = 5$, and $f(4) = 85$, so 0 is a minimum and 3 a maximum.

6. **(8 points)** *Estimate the following values using appropriate linear approximations.*

   (a) **(4 points)** $\sqrt[3]{1000.3}$.

   We observe that this number is very close to 1000, which has a nice cube root. We thus use a linear approximation to the function $f(x) = \sqrt[3]{x}$ near the point $a = 1000$. Since $f'(x) = \frac{1}{3x^{2/3}}$, we ascertain that $f(1000) = 10$ and $f'(1000) = \frac{1}{300}$ and so

   $$f(1000 + 0.3) \approx f(1000) + 0.3f'(1000) = 10 + \frac{0.3}{300} = 10.001$$

   Note that the actual value of $\sqrt[3]{1000.3}$ is approximately 10.0009999000166633396. so 10.001 is accurate to six decimal places.

   (b) **(4 points)** $(-2.994)^4$.

   We observe that this number is very close to -3, which has an easily calculated fourth power. We thus use a linear approximation to the function $f(x) = x^4$ near the point $a = -3$. Since $f'(x) = 4x^3$, we ascertain that $f(-3) = 81$ and $f'(-3) = 108$ and so

   $$f(-3 + 0.006) \approx f(-3) + 0.006f'(-3) = 81 - 0.648 = 80.352$$

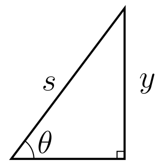
   Note that the actual value of $(-2.994)^4$ is approximately 80.353941409296. so 80.352 is a good but not great approximation.

7. **(12 points)** *Calculate* $\frac{d}{dt}\left[(t^3 + 5t^4)\tan(\ln t)\right]$.

   This expression is a product on its outermost level; we may observe that one of the terms in the product is a composition, so we will expect to use first the product rule, and then the chain rule.

   $$\begin{aligned}
   \frac{d}{dt}\left[\left(t^3 + 5t^4\right)\tan(\ln t)\right] &= \left[\frac{d}{dt}(t^3 + 5t^4)\right]\tan(\ln t) + (t^3 + 5t^4)\frac{d}{dt}\tan(\ln t) \\
   &= (3t^2 + 20t^3)\tan(\ln t) + (t^3 + 5t^4)\frac{d}{dt}\tan u \text{ where } u = \ln t \\
   &= (3t^2 + 20t^3)\tan(\ln t) + (t^3 + 5t^4)\frac{du}{dt}\frac{d}{du}\tan u \\
   &= (3t^2 + 20t^3)\tan(\ln t) + (t^3 + 5t^4)\frac{1}{t}\sec^2 u \\
   &= (3t^2 + 20t^3)\tan(\ln t) + \frac{(t^3 + 5t^4)\sec^2(\ln t)}{t}
   \end{aligned}$$

8. **(15 points)** *A parachuter, currently at a height of 0.4 miles above the ground, is falling straight downwards at a speed of 10 miles per hour. You are 0.3 miles away from the landing site, standing still and recording the descent with a camera.*



(a) **(9 points)** *How quickly are you and the parachuter approaching each other?*

The above image depicts the relationship in question; our distance from the landing site is a constant 0.3, while the parachuter's height and distance s are changing. From the given information, we know that $\frac{dy}{dt} = 10$, and we wish to know $\frac{ds}{dt}$. Since there is a right triangle with legs 0.3 and y and hypotenuse s, we also know that $0.3^2 + y^2 = s^2$. Differentiating both sides of this relationship with respect to time:

$$\begin{aligned}
\frac{d}{dt}\left(0.3^2 + y^2\right) &= \frac{d}{dt}s^2 \\
\frac{dy}{dt}\frac{d}{dy}\left(0.3^2 + y^2\right) &= \frac{ds}{dt}\frac{d}{ds}s^2 \\
\frac{y\frac{dy}{dt}}{s} &= \frac{ds}{dt}
\end{aligned}$$

and we know that y and $\frac{dy}{dt}$ are 0.4 and 10 at the moment; a little computation shows that $s = \sqrt{0.3^2 + 0.4^2} = 0.5$, and so $\frac{ds}{dt} = \frac{0.4 \cdot 10}{0.5} = 8$ miles per hour.

(b) **(6 points)** *How quickly should you be tilting the camera in order to keep the parachuter in the frame?*

We have the same scenario and labels as above, but now we seek to find $\frac{d\theta}{dt}$. There are several trigonometric relationships we can use to describe θ, but since y and 0.3 are the quantities in the triangle about which we know the most, the relationship $\tan\theta = \frac{y}{0.3}$ is particularly fruitful. Differentiating it with respect to time:

$$\begin{aligned}
\frac{d}{dt}\tan\theta &= \frac{d}{dt}\frac{y}{0.3} \\
\frac{d\theta}{dt}\frac{d}{d\theta}\tan\theta &= \frac{\frac{dy}{dt}}{0.3} \\
\frac{d\theta}{dt}\sec^2\theta &= \frac{\frac{dy}{dt}}{0.3} \\
\frac{d\theta}{dt} &= \frac{\frac{dy}{dt}\cos^2\theta}{0.3} = \frac{\frac{dy}{dt}\left(\frac{0.3}{s}\right)^2}{0.3} = \frac{10\frac{0.09}{0.25}}{0.3} = 12
\end{aligned}$$

Note that this result is in the unusual units of radians-per-hour; if you want a more palatable unit, this is about 11 degrees per minute.