No calculator is allowed for this test. For full credit show all of your work (legibly!), unless otherwise specified. Algebraic and trigonometric simplification of answers is generally unnecessary.

1. **(17 points)** Answer the following questions:

   (a) **(8 points)** *Find the general antiderivative of* $g(x) = \frac{x^3+3x+2}{x^2} - \frac{5}{\sqrt{1-x^2}} + \csc x \cot x$.
   The first term requires some cleanup; we may rewrite this expression as

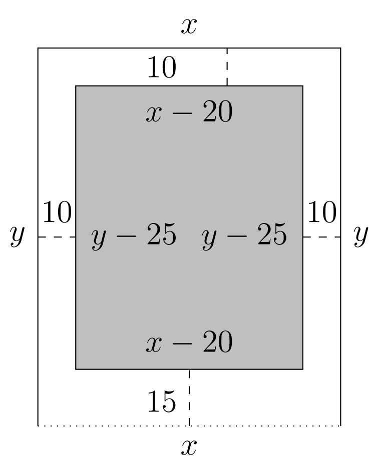
$$g(x) = x + 3x^{-1} + 2x^{-2} - \frac{5}{\sqrt{1-x^2}} + \csc x \cot x$$

   and then antidifferentiate term=by=term to find that

$$G(x) = \frac{1}{2}x + 3\ln|x| - 2x^{-1} - 5\arcsin x - \csc x + C$$

   (b) **(9 points)** *Given that* $f(2) = 0$ *and* $f'(t) = t^3 - \frac{16}{t^3}$, *find a formula for* $f(t)$.
   Using a general antiderviative, we know that $f(t) = \frac{t^4}{4} + \frac{8}{t^2} + C$. Then $f(2) = 4 + 2 + C$ and since we want $f(2)$ to be 0, we choose $C = -6$ to give the formula $f(t) = \frac{t^4}{4} + \frac{8}{t^2} - 6$.

2. **(24 points)** *An industrial manufacturer has budgeted 8000 square feet of floor space for a rectangular factory with a loading dock running along one side. They want to fill it as full as possible with machines, but regulations require 10 feet of open floor space along each wall, and 15 feet of open floor space along the loading dock. What dimensions for the factory will maximize the quantity of floor space available for their machines?*



The above drawing is a representation of the scenario described; we assign the two dimensions of the factory the labels of x and y (we could alternatively label the dimensions of the usable area with x and y, which would give correct results, but it would make the arithmetic a bit messier). Since there is a gap of 10 feet along each wall and 15 feet along the loading dock, the height of the usable area will be 25 feet less than the area of the factory; likewise, the width of the usable space is 20 feet less than the width of the factory, so the usable region is an $(x - 20) \times (y - 25)$ rectangle.

Our constraint is that the factory as a whole has an area of 8000 square feet, so we are constrained that the non-negative quantities x and y must be such that $xy = 8000$. What we seek to maximize is the usable area $(x-20)(y-25)$; rephrasing the above constraint as $y = \frac{8000}{x}$, we see that the area is

$$A(x) = (x-20)\left(\frac{8000}{x} - 25\right) = 8000 - 25x - \frac{160000}{x} + 500 = 850 - 25x - \frac{160000}{x}$$

Our limits of x are dictated by the need for x to be at least 20 to permit the wall clearance, and for y to be at least 25 to permit wall-and-dock clearance. Thus $x \geq 20$ and $\frac{8000}{x} \geq 25$, so $20 \leq x \leq 320$.

Solving this maximization problem, we observe that $A'(x) = -25 + \frac{160000}{x^2}$. This is undefined when $x = 0$, and is zero when $x^2 = 6400$, or when $x = \pm 80$. We note that $x = 0$ and $x = -80$ are outside our interval, leaving us with the potential optima $x = 20$, $x = 80$, and $x = 320$. Unsurprisingly, $A(20) = 0$ and $A(320) = 0$, since both of these are trivial factories which are either too narrow or too short to contain any machines, and we are left with an 80×100 factory as optimal (which has a 60×75 floor space suitable for placing 4500 square feet of machinery).

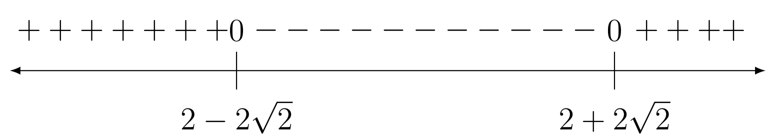
3. **(24 points)** *Answer the following questions related to the shape of the graph of the function* $f(x) = x^3 - 6x^2 - 12x + 5$.

   (a) **(4 points)** *What are $f(x)$'s long term behaviors as x grows very large and as x grows very negative? Describe each direction in either words or symbols.*

   Since the dominant term in $f(x)$ is x^3, we kow that $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^3$, so that as x gets very large, so does $f(x)$, and as x gets very negative, so does $f(x)$.

   (b) **(6 points)** *Where is $f(x)$ increasing? Where is it decreasing? Label which is which.*

   This answer is dependent on the sign of $f'(x)$ at various values. We know that $f'(x) = 3x^2 - 12x - 12 = 3(x^2 - 4x - 4)$, which is zero when $x = \frac{4\pm\sqrt{32}}{2} = 2 \pm 2\sqrt{2}$ (this problem was poorly designed on that front; ordinarily I try for rational roots). We can probe at $x = 2$, $x = 5$, and $x = -1$ to look at the regions around the zeroes: note that $f'(-1) = 3$, $f'(2) = -36$, and $f'(5) = 3$, so the sign of $f'(x)$ behaves as such:



   Thus $f(x)$ is increasing when $x < 2 - 2\sqrt{2}$ and $x > 2 + 2\sqrt{2}$, and is decreasing when $2 - 2\sqrt{2} < x < 2 + 2\sqrt{2}$.

   (c) **(6 points)** *What are its critical points, and is each a local maximum, a local minimum, or neither?*

   As seen above, this function achieves a zero derivative, and thus criticality, at $x = 2\pm 2\sqrt{2}$. At $x = 2 - 2\sqrt{2}$ it transitions from increasing to decreasing, and thus this point is a local maximum; at $x = 2 + 2\sqrt{2}$ it changes from decreasing to increasing, which denotes a local minimum.

   (d) **(8 points)** *Where is it concave up? Where is it concave down? Label which is which. Where, if anywhere, are its points of inflection?*

   This answer is dependent on the sign of $f''(x)$ at various values. We know that $f''(x) = 6x - 12$, which is zero when $x = 2$, positive when $x > 2$, and negative when $x < 2$. Thus

$f(x)$ is concave up when $x > 2$, concave down when $x < 2$, and possesses a point of inflection at $x = 2$.

4. **(12 points)** *Answer the following questions about approximation with Newton's method:*

   (a) **(6 points)** *Starting with an initial value of 3, use two iterations of Newton's method to approximate a zero of $f(x) = x^3 - 4x^2 - 2x + 14$. Your answer need not be arithmetically simplified.*
   Recall that Newton's method is the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. In this case, where our function is $x^3 - 4x^2 - 2x + 14$, the formula will be specifically $x_{n+1} = x_n - \frac{x_n^3 - 4x_n^2 - 2x_n + 4}{3x_n^2 - 8x_n - 2}$. Running through two iterations:

$$x_1 = 3$$
$$x_2 = 3 - \frac{3^3 - 4 \cdot 3^2 - 2 \cdot 3 + 4}{3 \cdot 3^2 - 8 \cdot 3 - 2} = 3 - \frac{-11}{1} = 14$$
$$x_3 = 14 - \frac{14^3 - 4 \cdot 14^2 - 2 \cdot 14 + 4}{3 \cdot 14^2 - 8 \cdot 14 - 2} = \frac{2350}{237} \approx 9.9156$$

   After about 8 iterations, this approach settles down somewhat to a solution at approximately 4.249143.

   (b) **(6 points)** *Choose $x_1 = 4$ to be an initial approximation of $\sqrt{13}$. Use one step of Newton's method on an appropriately chosen polynomial function to develop x_2, a better rational approximation of $\sqrt{13}$; also give an arithmetic expression (which need not* **and probably should not** *be simplified) for the better approximation x_3 arising from a second step of Newton's method.*
   $\sqrt{13}$ is a root of the polynomial $x^2 - 13$ (there are others, but this is the most obvious choice), so we want to use Newton's method specifically incarnated as the rule $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^2-13}{2x}$. Running this twice:

$$x_1 = 4$$
$$x_2 = 4 - \frac{4^2 - 13}{2 \cdot 4} = 4 - \frac{3}{8} = \frac{29}{8} = 3.625$$
$$x_3 = \frac{29}{8} - \frac{\left(\frac{29}{8}\right)^2 - 13}{2\left(\frac{29}{8}\right)^2} = \frac{1673}{464} \approx 3.605603$$

   This is not a bad estimator for the actual value of $\sqrt{13}$, which is approximately 3.6055513.

5. **(23 points)** *Evaluate the following limits; if they cannot be evaluated, show why not.*

   (a) $\lim_{\theta \to 0} \frac{\cos\theta}{e^\theta + 2}$.
   This limit may be directly evaluated to get a result of $\frac{\cos 0}{e^0 + 2} = \frac{1}{3}$.

   (b) $\lim_{q \to -\infty} q e^q$.
   This limit would appear to be a $\infty \cdot 0$ indeterminate form on direct evaluation, so it must be recast as a fraction. The most straightforward way to do so is to rewrite it as

$\lim_{q\to-\infty} \frac{q}{e^-q}$, which is a $\frac{\infty}{\infty}$ indeterminate form. Using L'Hôpital's rule,

$$\lim_{q\to-\infty} \frac{q}{e^-q} = \lim_{q\to-\infty} \frac{1}{-e^-q}$$

which has the form $\frac{1}{\infty}$, which tends towards zero, achieving a limit value of 0.

(c) $\lim_{t\to+\infty} \frac{e^t}{t^2 \ln t}$.

This limit is a $\frac{\infty}{\infty}$ indeterminate form, so we invoke L'Hôpital's rule (mindful to use the product rule on the denominator):

$$\lim_{t\to+\infty} \frac{e^t}{t^2 \ln t} = \lim_{t\to+\infty} \frac{e^t}{2t \ln t + \frac{t^2}{t}} = \lim_{t\to+\infty} \frac{e^t}{2t \ln t + t}$$

but this remains a $\frac{\infty}{\infty}$ form, so we apply L'Hôpital's rule again:

$$\lim_{t\to+\infty} \frac{e^t}{2t \ln t + t} = \lim_{t\to+\infty} \frac{e^t}{2 \ln t + 2\frac{2t}{t} + 1} = \lim_{t\to+\infty} \frac{e^t}{2 \ln t + 3}$$

and finally, since this is still a $\frac{\infty}{\infty}$ form, one more time:

$$\lim_{t\to+\infty} \frac{e^t}{2 \ln t + 3} = \lim_{t\to+\infty} \frac{e^t}{\frac{2}{t}} = \lim_{t\to+\infty} \frac{te^t}{2}$$

and this is no longer indeterminate, although it is infinite, so this limit does not exist (specifically, increasing without bound).

(d) $\lim_{u\to 0} \frac{\sin u - u}{u^3}$.

This limit is a $\frac{0}{0}$ indeterminate form, so we apply L'Hôpital's rule:

$$\lim_{u\to 0} \frac{\sin u - u}{u^3} = \lim_{u\to 0} \frac{\cos u - 1}{3u^2}$$

which still evaluates to $\frac{0}{0}$, so we apply it again:

$$\lim_{u\to 0} \frac{\cos u - 1}{3u^2} = \lim_{u\to 0} \frac{-\sin u}{6u}$$

and once more, since this is still $\frac{0}{0}$:

$$\lim_{u\to 0} \frac{-\sin u}{6u} = \lim_{u\to 0} \frac{-\cos u}{6} = \frac{-1}{6}$$

(e) $\lim_{x\to 0} \frac{x-\sqrt{x}}{e^x-1}$.

This limit is a $\frac{0}{0}$ indeterminate form, so we apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{x-\sqrt{x}}{e^x-1} = \lim_{x\to 0} \frac{1-\frac{1}{2\sqrt{x}}}{e^x}$$

and now, an attempt at evaluation gives division by zero, but no indeterminate forms, resulting in a limit at a "blow-up" point (i.e. a vertical asymptote) which thus does not exist.

6. **(6 point bonus)** *Prove that if $f(x)$ is a function which is continuous with a continuous derivative and k local minima, it must have between $k-1$ and $k+1$ local maxima.*