

1. **(15 points)** Answer the following questions.

(a) **(5 points)** What is the general antiderivative of  $\frac{x^2-1}{x^2+1} - e^x + \csc x(\csc x + \cot x)$ ?

The first term needs simplification, and the third term needs expansion; note that  $\frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$ , which is more conducive to antidifferentiation. So now we wish to take the antiderivative of  $1 - \frac{2}{x^2+1} - e^x + \csc^2 x + \csc x \cot x$ , and identifying these known forms termwise we get an antiderivative of  $x - 2 \arctan x - e^x - \cot x - \csc x + C$ .

(b) **(5 points)** If  $f''(x) = 6x$ ,  $f'(2) = 3$ , and  $f(1) = 0$ , find a formula for  $f(x)$ .

Taking the antiderivative of  $f''(x)$  gives us  $f'(x)$ , so  $f'(x) = 3x^2 + C$ . Since  $f'(2) = 3$ , we will require that  $3 \cdot 2^2 + C = 3$ , or  $C = -9$ ; thus  $f'(x) = 3x^2 - 9$ . Taking the antiderivative again to get  $f(x)$ , we find that  $f(x) = x^3 - 9x + D$ , and since  $f(1) = 0$ , we require that  $1 - 9 + D = 0$ , or that  $D = 8$ . Thus,  $f(x) = x^3 - 9x + 8$ .

(c) **(6 points)** Simplify the expression  $\frac{d}{dx} \int_{x^2}^{x^3} \ln(t^2 - 7) dt$ .

We do not know the antiderivative of  $\ln(t^2 - 7)$ ; let's simply call it  $F(t)$ . We'll never need to know what it actually is, because the antiderivative and derivative will hopefully cancel out. We then invoke the Fundamental Theorem of Calculus to write the definite integral in terms of  $F(t)$ :

$$\frac{d}{dx} \int_{x^2}^{x^3} \ln(t^2 - 7) dt = \frac{d}{dx} (F(x^3) - F(x^2))$$

and then use the chain rule to differentiate those two applications of  $F$ :

$$\frac{d}{dx} (F(x^3) - F(x^2)) = 3x^2 F'(x^3) - 2x F'(x^2)$$

and finally, since  $F(t)$  is the antiderivative of  $\ln(t^2 - 7)$ ,  $F'(t) = \ln(t^2 - 7)$ , and thus:

$$3x^2 F'(x^3) - 2x F'(x^2) = 3x^2 \ln(x^6 - 7) - 2x \ln(x^4 - 7)$$

2. **(20 points)** You have 1200 square centimeters of material with which to make a box with a square base and an open top. Find the dimensions which maximize the volume of the box.

Let us denote the dimensions of the base in centimeters (which are equal, since the base is a square) as  $x$ , and let the height of the box be  $h$  centimeters. Then the box uses a total of  $x^2 + 4xh$  square centimeters of material (since the base is of area  $x^2$ , and the four lateral faces are each of area  $xh$ ), and has a volume of  $x^2h$ .

Thus, our material limitations require that  $x^2 + 4xh = 1200$ , and our goal is to maximize  $x^2h$ . Solving the constraint for  $h$ , we find that  $h = \frac{1200-x^2}{4x}$ , so our volume, written solely in terms of  $x$  is the function  $V(x) = x^2 \left( \frac{1200-x^2}{4x} \right)$ . We seek to maximize this function on the interval  $(0, \sqrt{1200})$  — whose lower bound comes from the fact that  $x$  must be positive as it is a length, and whose upper bound comes from the fact that there must be at least a little left over material left to build the lateral faces, so  $x^2 < 1200$ .

Solving this problem in a traditional manner, we see that  $V(x) = 300x - \frac{x^3}{4}$ , so  $V'(x) = 300 - \frac{3x^2}{4}$ . This is continuous everywhere, so its only critical points are where  $300 - \frac{3x^2}{4} = 0$ , which can be algebraically converted to  $x^2 = 400$ , so  $x = \pm 20$ .

The critical point  $x = -20$  can be rejected as being outside of the interval; the remaining critical point and the two ends of the interval are tested:

$$\lim_{x \rightarrow 0^+} V(x) = V(0) = 0$$

$$V(20) = 20^2 \left( \frac{1200 - 400}{80} \right) = 4000$$

$$\lim_{x \rightarrow \sqrt{1200}^-} V(x) = V(\sqrt{1200}) = 0$$

so our optimal dimensions are  $20 \times 20 \times 10$ .

3. **(15 points)** *The lemniscate is a curve satisfying the equation  $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ .*

(a) **(10 points)** *Find a formula for  $\frac{dy}{dx}$  on this curve in terms of  $x$  and  $y$ .*

We proceed by implicit differentiation:

$$\begin{aligned} \frac{d}{dx} [2(x^2 + y^2)^2] &= \frac{d}{dx} [25(x^2 - y^2)] \\ 4(x^2 + y^2) \frac{d}{dx} (x^2 + y^2) &= 25 \frac{d}{dx} (x^2 - y^2) \\ 4(x^2 + y^2) \left( 2x + 2y \frac{dy}{dx} \right) &= 25 \left( 2x - 2y \frac{dy}{dx} \right) \\ 8x^3 + 8x^2y \frac{dy}{dx} + 8xy^2 + 8y^3 \frac{dy}{dx} &= 50x - 50y \frac{dy}{dx} \\ 8x^2y \frac{dy}{dx} + 8y^3 \frac{dy}{dx} + 50y \frac{dy}{dx} &= 50x - 8x^3 - 8xy^2 \\ \frac{dy}{dx} &= \frac{50x - 8x^3 - 8xy^2}{8x^2y + 8y^3 + 50y} \end{aligned}$$

(b) **(5 points)** *Find the equation of the tangent line to the lemniscate at the point  $(3, 1)$ .*

Using the above result, we determine that at  $(3, 1)$ ,

$$m = \frac{dy}{dx} = \frac{150 - 216 - 24}{72 + 8 + 50} = \frac{-9}{13}$$

so using the point-slope formula, we have an equation of  $(y - 1) = \frac{-9}{13}(x - 3)$ .

4. **(18 points)** *Evaluate the following integrals:*

(a) **(6 points)**  $\int_0^4 x\sqrt{16 - x^2} dx$ .

We make use of a substitution for  $16 - x^2$ , yielding the equation  $u = 16 - x^2$  and the pseudo-equation  $du = -2x dx$ . Thus  $x dx = \frac{-1}{2} du$ , which we may use to convert the entire integral, and then evaluate it:

$$\int_0^4 x\sqrt{16 - x^2} dx = \int_{16}^0 -\frac{1}{2}\sqrt{u} du = -\frac{1}{3}u^{3/2} \Big|_{16}^0 = -\frac{0^{3/2}}{3} + \frac{16^{3/2}}{3} = \frac{64}{3}$$

(b) **(6 points)**  $\int \frac{x^3}{x^4+1} dx$ .

We shall substitute  $u$  for  $x^4 + 1$ , yielding the pseudo-equation  $du = 4x^3 du$ , so we may convert our integral as such:

$$\int \frac{x^3}{x^4+1} dx = \int \frac{1}{4u} du = \frac{1}{4} \ln |u| + C = \frac{1}{4} \ln |x^4 + 1| + C$$

(c) **(6 points)**  $\int_0^{\pi/3} \cos \theta + 2 \sec \theta \tan \theta d\theta$ .

This is an integral which can be directly evaluated:

$$\int_0^{\pi/3} \cos \theta + 2 \sec \theta \tan \theta d\theta = \left[ \sin \theta + 2 \sec \theta \right]_0^{\pi/3} = \sin \frac{\pi}{3} + 2 \sec \frac{\pi}{3} - \sin 0 - 2 \sec 0 = \frac{\sqrt{3}}{2} + 4 - 0 - 2 = 2 + \frac{\sqrt{3}}{2}$$

5. **(16 points)** A ten-foot-long ladder is leaning against a wall, with the base of the ladder six feet from the wall. The base is sliding away from the wall at a rate of half a foot per hour.

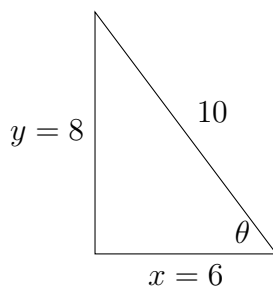
(a) **(8 points)** How quickly is the top of the ladder slipping down the wall?

Let us denote the height of the ladder up the wall by  $y$ , and the distance of the base from the wall by  $x$ . Under this naming scheme, the desired quantity is  $\frac{dy}{dt}$ . Then by the Pythagorean Theorem,  $x^2 + y^2 = 10^2$  (since the length of the ladder is constant). We also know that  $x$  is currently 6, and can deduce that  $y$  is currently  $\sqrt{10^2 - 6^2} = 8$ . In addition, the problem tells us that  $\frac{dx}{dt} = 0.5$ . Thus:

$$\begin{aligned} \frac{d}{dt}(x^2 + y^2) &= \frac{d}{dt}100 \\ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= \frac{-x \frac{dx}{dt}}{y} = \frac{-3}{8} \end{aligned}$$

so the ladder is sliding down the wall at three-eighths of a foot per hour.

(b) **(8 points)** How quickly is the angle between the ladder and the floor changing?



Let us use the same variables as above, and additionally let  $\theta$  represent the angle between the ladder and the floor, so that our quantity of interest here is  $\frac{d\theta}{dt}$ . Since  $\theta$  is an angle of a right triangle, we can express its trig functions as ratio of the sides; the most usable formula here is  $\cos \theta = \frac{x}{10}$ , although others will also work. Taking the derivative of both sides, we see that

$$\frac{d\theta}{dt}(-\sin \theta) = \frac{dx}{dt} \frac{1}{10}$$

and thus that

$$\frac{d\theta}{dt} = -\frac{dx}{dt} \csc \theta = \frac{-0.5}{10} \cdot \frac{10}{8} = \frac{-1}{16}$$

6. (16 points) Compute the following expressions:

(a) (6 points) Compute  $\frac{d}{dx}\sqrt{\arctan \sqrt{x}}$ .

In preparation for the chain rule, let us denote  $\sqrt{\arctan \sqrt{x}}$  as  $\sqrt{u}$ , where  $u = \arctan v$  and  $v = \sqrt{x}$ . Then

$$\frac{d}{dx}\sqrt{u} = \frac{du}{dx} \frac{1}{2\sqrt{u}} = \frac{dv}{dx} \frac{du}{dv} \frac{1}{2\sqrt{u}}$$

and evaluating all these known derivatives, we get a result of

$$\frac{1}{2\sqrt{x}} \frac{1}{1+v^2} \frac{1}{2\sqrt{u}} = \frac{1}{4\sqrt{x}(1+x)\sqrt{\arctan \sqrt{x}}}$$

(b) (6 points) Given  $f(t) = \tan \frac{e^t}{\arcsin t}$ , find  $f'(t)$ .

Let  $u = \frac{e^t}{\arcsin t}$ ; using the quotient rule we may find that  $\frac{du}{dt} = \frac{\arcsin t e^t - e^t \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2}$ . Then, using the chain rule:

$$f'(t) = \frac{d}{dt} \tan u = \frac{du}{dt} \sec^2 u = \frac{\arcsin t e^t - e^t \frac{1}{\sqrt{1-t^2}}}{(\arcsin t)^2} \sec^2 \frac{e^t}{\arcsin t}$$

(c) (4 points) Find  $\int \frac{d}{ds} \frac{s^3}{\sqrt{s^2+5}} ds$ .

This is an antiderivative of a derivative, and it is known that doing so simply introduces a constant, so the answer is  $\frac{s^3}{\sqrt{s^2+5}} + C$ .

7. (12 points) Consider the function  $g(x) = \frac{x}{x^2+9}$ .

(a) (5 points) Identify the zeroes, vertical asymptotes, and long-term behavior on both sides of this function. Clearly label which is which, and if any features are not present, say so.

The numerator is zero (and thus the function is zero) when  $x = 0$ ; vertical asymptotes appear when the denominator is zero, but that never happens in this case, so this function has no vertical asymptotes. As  $x$  grows very large or very negative, this function behaves like  $\frac{x}{x^2} = \frac{1}{x}$ , which tends towards zero in both directions.

(b) (5 points) Identify the critical points of this function, and indicate whether each is a local maximum, local minimum, or neither.

Note that  $g'(x) = \frac{(x^2+9)-2x^2}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$ . This exists everywhere, since the denominator is never zero, but the numerator is zero at  $x = \pm 3$ ; thus this function has critical points at  $x = \pm 3$ . Since  $g'(-4) = g'(4)$  is negative and  $g'(0)$  is positive, we see that  $x = -3$  is a minimum and  $x = +3$  is a maximum.

(c) (2 points) Which if any of the critical points identified above are global maxima or global minima? Show work or explain.

Both are global extrema: this function has no asymptotes or long-term behavior that could “spil” them, so they are in fact the function’s global minimum and maximum.

8. (15 points) Determine the following limits.

(a) (5 points) Evaluate  $\lim_{\theta \rightarrow 0} \frac{\theta - \sin \theta}{\theta - \tan \theta}$  or demonstrate that it cannot be evaluated.

Since  $0 - \sin 0 = 0$  and  $0 = \tan 0 = 0$ , this is a  $\frac{0}{0}$  indeterminate form. We may thus invoke L'Hôpital's rule:

$$\lim_{\theta \rightarrow 0} \frac{\theta - \sin \theta}{\theta - \tan \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{1 - \sec^2 \theta}$$

which is still a  $\frac{0}{0}$  form, so we continue:

$$\lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{1 - \sec^2 \theta} = \lim_{\theta \rightarrow 0} \frac{-\sin \theta}{-2 \sec^2 \theta \tan \theta} = \lim_{\theta \rightarrow 0} \cos^3 \theta = \frac{1}{2}$$

- (b) **(5 points)** Evaluate  $\lim_{t \rightarrow 0} \frac{e^t - 1 - t}{t^2}$  or demonstrate that it cannot be evaluated.

This is a  $\frac{0}{0}$  indeterminate form, so we may use L'Hôpital's rule:

$$\lim_{t \rightarrow 0} \frac{e^t - 1 - t}{t^2} = \lim_{t \rightarrow 0} \frac{e^t - 1}{2t}$$

and again, as it is still  $\frac{0}{0}$ :

$$\lim_{t \rightarrow 0} \frac{e^t - 1}{2t} = \lim_{t \rightarrow 0} \frac{e^t}{2} = \frac{1}{2}$$

- (c) **(5 points)** Using the difference quotient, find the derivative with respect to  $x$  of the function  $f(x) = 20 + 3x - 5x^2$ . You may not use L'Hôpital's rule for this problem.

$$\begin{aligned} f'(x) &= \lim_{x \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{x \rightarrow 0} \frac{(20 + 3(x+h) - 5(x+h)^2) - (20 + 3x - 5x^2)}{h} \\ &= \lim_{x \rightarrow 0} \frac{3h - 10xh - 5h^2}{h} \\ &= \lim_{x \rightarrow 0} 3 - 10x - 5h = 3 - 10x \end{aligned}$$

9. **(15 points)** Let  $f(x) = 2 + 2x^2 - x^4$ .

- (a) **(5 points)** Where is  $f(x)$  increasing? Where is it decreasing? Label which is which.

We consider the sign-changes of  $f'(x) = 4x - 4x^3 = -4x(x+1)(x-1)$ . This is clearly zero at  $x = -1$ ,  $x = 1$ , and  $x = 0$ . Probing between and among the zeroes indicates that when  $x < -1$ ,  $f'(x)$  is positive, when  $-1 < x < 0$ ,  $f'(x)$  is negative, when  $0 < x < 1$ ,  $f'(x)$  is positive, and when  $x > 1$ ,  $f'(x)$  is negative again. Thus  $f(x)$  is decreasing on  $(-1, 0)$  and  $(1, \infty)$ , and is increasing on  $(-\infty, -1)$  and  $(0, 1)$ .

- (b) **(3 points)** What are the critical points of  $f(x)$ ? Is each a local maximum, a local minimum, or neither?

Since  $x = -1$  is a transition from increase to decrease, it is a local maximum, since  $x = 0$  is a transition from decrease to increase, it is a local minimum, and since  $x = 1$  is a transition from increase to decrease, it is a local maximum.

- (c) **(7 points)** Determine where  $f(x)$  is concave up and where it is concave down, and identify points of inflection.

We consider the sign-changes of  $f''(x) = 4 - 12x^2$ . This is zero at  $x = \pm \frac{1}{\sqrt{3}}$ , and probing to the left and right of these zeroes indicates that  $f''(x) < 0$  when  $|x| > \frac{1}{\sqrt{3}}$ , and  $f''(x) > 0$  when  $|x| < \frac{1}{\sqrt{3}}$ . Thus,  $f(x)$  is concave down on both  $(-\infty, -\frac{1}{\sqrt{3}})$  and  $(\frac{1}{\sqrt{3}}, \infty)$ , and concave up on  $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ . The transitions at  $x = \pm \frac{1}{\sqrt{3}}$  will be points of inflection.

10. **(8 points)** Answer the following questions about the function  $f(x) = \frac{x^{2/3}}{x+1}$ .

(a) **(4 points)** What is the domain of  $f(x)$ ?

The numerator is not a problem; cube roots are valid anywhere. However, the denominator is zero when  $x = -1$ , so the domain is  $x \neq -1$ .

(b) **(4 points)** Where does the derivative of  $f(x)$  exist?

$f'(x) = \frac{(x+1) \cdot \frac{2}{3}x^{-1/3} - x^{2/3}}{(x+1)^2}$ . This still has problems at  $x = -1$ , but additionally it is problematic at  $x = 0$ , since 0 cannot be raised to a negative power. Thus, this function is differentiable only when  $x$  is neither 0 nor  $-1$ .