1. **(5 points)** *Using linear approximation techniques, find a good rational estimate for the value of* $\sqrt[4]{15.9984}$.

   Let $f(x) = \sqrt[4]{x} = x^{1/4}$ so that $f'(x) = \frac{1}{4x^{3/4}}$; then $f(16) = 2$ and $f'(16) = \frac{1}{4 \cdot 8} = \frac{1}{32}$. Now we may apply a linear approximation:

   $$f(15.9984) = f(16 - 0.0016) \approx f(16) - 0.0016 f'(16) = 2 - \frac{0.0016}{32} = 1.99995$$

   The actual value of $\sqrt[4]{15.9984}$ is $1.99994999812489061747\ldots$; note that our result is accurate to within eight decimal places!

2. **(6 points)** *Find the values of* x *which minimize and maximize the function* $f(x) = x^3 - 9x^2 - 21x + 10$ *on the interval* $[-2, 2]$.

   $f'(x) = 3x^2 - 18x - 21 = 3(x^2 - 6x - 7) = 3(x-7)(x+1)$, so $f'(x)$ exists everywhere and is zero when $x = 7$ or $x = -1$. Thus the critical points of this function are $x = 7$ and $x = -1$; the former we reject as a candidate for interval extremum because it is outside our interval. Thus, the only candidates for extrema in the interval are the endoints -2 and 2, and the critical point -1. $f(-2) = -8 - 36 + 42 + 10 = 8$, $f(2) = 8 - 36 - 42 + 10 = -60$, and $f(-1) = -1 - 9 + 21 + 10 = 21$. Thus, a minimum occurs at $x = 2$ and a maximum at $x = -1$

3. **(8 points)** *Answer the following questions about the function* $f(x) = x^4 - 8x^3 + 18x^2 - 2$.

   (a) **(4 points)** *On what intervals is it increasing, and on what intervals is it decreasing?*

   $f'(x) = 4x^3 - 24x^2 + 36x = 4(x^3 - 6x^2 + 9x) = 4x(x-3)^2$. We may thus be certain that $f'(x)$ exists and is continuous everywhere, and it is zero at $x = 0$ and $x = 3$. We may now probe in the three intervals $(-\infty, 0)$, $(0, 3)$, and $(3, \infty)$ to determine where $f'(x)$ is positive and where negative.

   Probing $f'(-1) = 4(-1 - 6 - 9) = -64$, so $f'(x)$ is negative on $(-\infty, 0)$. Probing $f'(1) = 4(1-6+9) = 16$, we see that $f'(x)$ is positive on $(0, 3)$. Probing $f'(4) = 4(64-96+36) = 16$, we see that $f'(x)$ remains positive on $(3, \infty)$. Thus $f(x)$ is decreasing on $(-\infty, 0)$, and increasing on $(0, 3)$ and $(3, \infty)$.

   (b) **(2 points)** *What* x*-values are local extrema of this function, and what type of extremum is each?*

   Since, as seen above, $f(x)$ transitions from decrease to increase at $x = 0$, the point $x = 0$ is a local minimum. Although $x = 3$ is a critical point (and is thus under consideration as an extremum), it is not actually a local extremum, since both before and after $x = 3$ the function is increasing.

   (c) **(3 points)** *On what intervals is this function concave up, and on which intervals is it concave down? Where are the points of inflection?*

   $f''(x) = 12x^2 - 48x + 36 = 12(x^2 - 4x + 3) = 12(x-3)(x-1)$, so $f''(x)$ exists everywhere and is zero at $x = 3$ and $x = 1$. We probe in the three intervals so found to determine concavity: in $(-\infty, 1)$, we might test $f''(0) = 12 \cdot 3 = 36$ to find that $f''(x)$ is positive, in $(1, 3)$ we might test $f''(2) = 12 \cdot -1 = -12$ to find that $f''(x)$ is negative, and in $(3, \infty)$ we might test $f''(4) = 12 \cdot 3 = 36$ to find that $f''(x)$ is positive. Thus, $f(x)$ is concave up on $(-\infty, 1)$, concave down on $(1, 3)$, and concave up again on $(3, \infty)$. The two points of transition at $x = 1$ and $x = 3$ are points of inflection.

4. **(2 point bonus)** *On the back of this sheet, give a formula for a continuous function with infinitely many local minima, infinitely many local maxima, exactly one global maximum, and no global minimum. Justify your assertion.*