1. **(10 points)** *Evaluate the following limits:*

   (a) **(4 points)** $\lim_{\theta\to 0} \theta \cot(2\theta)$.

   It is a truism that as $\theta \to 0$, $\theta \to 0$. Less intuitively, as $\theta \to 0$, $\cot(2\theta) \to \pm\infty$, so this limit is a $0 \cdot \infty$ indeterminate form. A simple way to recast it would be to write it as the quotient $\lim_{\theta\to 0} \frac{\theta}{\tan(2\theta)}$, which is a $\frac{0}{0}$ indeterminate form suitable for applying L'Hôpital's rule. Then

   $$\lim_{\theta\to 0} \frac{\theta}{\tan(2\theta)} = \lim_{\theta\to 0} \frac{1}{2\sec^2(2\theta)}$$

   and this latter limit directly evaluates to $\frac{1}{2}$.

   Note that it is also possible to rewrite the above limit as $\lim_{\theta\to 0} \frac{\theta\cos(2\theta)}{\sin(2\theta)}$, which is also a $\frac{0}{0}$ form, and which we apply L'Hôpital's rule to and get $\lim_{\theta\to 0} \frac{\cos(2\theta)-2\theta\sin 2\theta}{2\cos(2\theta)} = \frac{1}{2}$.

   (b) **(2 points)** $\lim_{t\to 4} \frac{\sqrt{t}+2}{t^2+16}$

   L'Hôpital's rule is emphatically not called for here: direct evaluation gives $\frac{4}{32} = \frac{1}{8}$.

   (c) **(4 points)** $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

   (d) $\cos 0 - 1 = 0$ and $0^2 = 0$, so the above limit is a $\frac{0}{0}$ form, to which we can apply L'Hôpital's rule:

   $$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x}$$

   which is unfortunately still a $\frac{0}{0}$ form, so we use L'Hôpital's rule again:

   $$\lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2}$$

   which evaluates to $\frac{-1}{2}$.

2. **(10 points)** *Answer the following questions which would be preparatory to sketching the graph of the function $y = (x+3)e^x$.*

   (a) **(2 points)** *What is its long-term behavior in each direction? You may describe it in words or symbolically.*

   As x gets very large, both $x+3$ and e^x get very large, so $\lim_{x\to+\infty}(x+3)e^x = +\infty$.

   As x gets very negative, $x+3$ also becomes very negative while e^x approaches zero, so $\lim_{x\to-\infty}(x+3)e^x$ is a $\infty \cdot 0$ indeterminate form. If we rearragine it into the form $\lim_{x\to-\infty} \frac{x+3}{e^{-x}}$, L'Hôpital's rule yields $\lim_{x\to-\infty} \frac{1}{-e^-x} = \lim_{x\to-\infty} -e^x = 0$.

   (b) **(4 points)** *On which intervals is it increasing, and on which intervals is it decreasing? Label which is which.*

   Using the product rule, $y' = 1e^x + (x+3)e^x = (x+4)e^x$. This is positive when $x > -4$ and negative when $x < -4$, so the graph is increasing for $x > -4$ and decreasing for $x < -4$ (with a minimum at $x = -4$, incidentally).

   (c) **(4 points)** *On which intervals is it concave up, and on which intervals is it concave down? Label which is which.*

   Using the product rule, $y'' = 1e^x + (x+4)e^x = (x+5)e^x$. This is positive when $x > -5$ and negative when $x < -5$, so the graph is concave up for $x > -5$ and concave down for $x < -5$ (with a point of inflection at $x = -5$, incidentally).

3. **(2 point bonus)** *Calculate* $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$ *as an expression in* a *and* b.